Explicit Pieri Inclusions

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Integer partitions

Definition

A **partition** of $n \in \mathbb{N}$ is a sequence of integers

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum \lambda_i = n$.

- Only finitely many $\lambda_i \neq 0$
- Each $\lambda_i \neq 0$ is a *part* of λ
- $\lambda \vdash n$ or $|\lambda| = n$

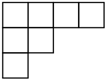
$$\lambda = (4, 2, 1, 0, 0, \ldots) = (4, 2, 1, 0, 0) = (4, 2, 1) \qquad |\lambda| = 7$$

Young diagrams

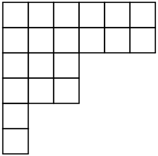
Partitions of n can be represented by a *Young diagram* of size n, an array of n left-justified boxes with weakly decreasing row length.

Examples:

$$\lambda = (4, 2, 1) \qquad \longleftrightarrow \qquad \lambda =$$



$$\sigma = (6, 6, 3, 3, 1, 1) \qquad \longleftrightarrow \qquad \sigma =$$



Blocks of a diagram

Block notation: $\lambda = (w_1^{h_1}, w_2^{h_2} \ldots, w_{N-1}^{h_{N-1}}, w_N^{h_N})$ where $w_i < w_{i+1}$ and each w_i appears as a part of λ exactly h_i times.

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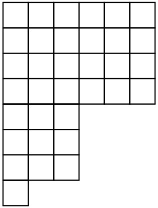

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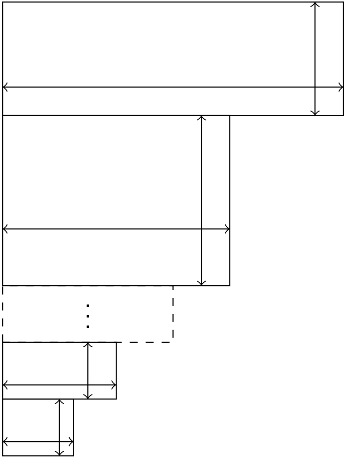

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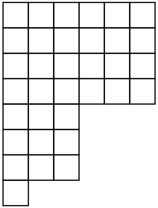

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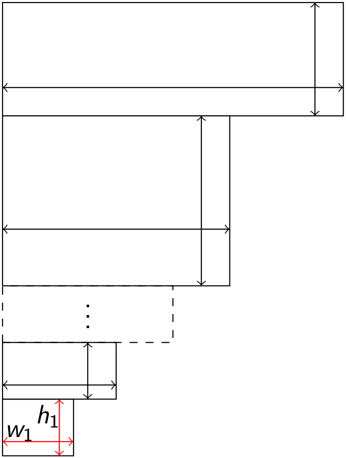

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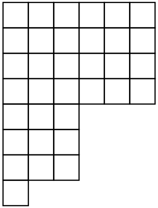

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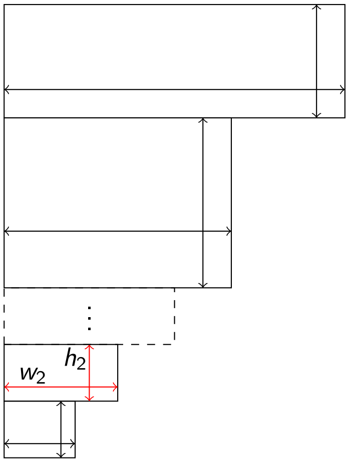

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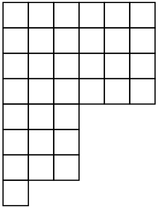

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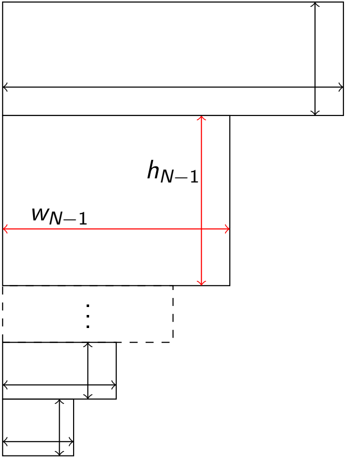

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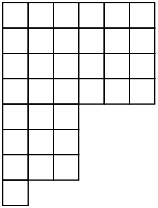

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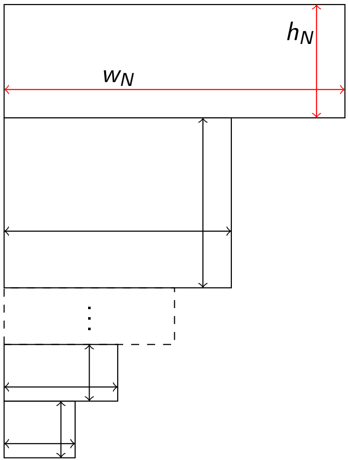

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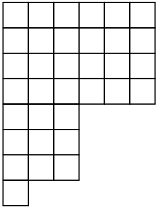

Young tableaux

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4	2	9	1
7	5		
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Semi-standard ones are weakly increasing across rows and strictly increasing down columns.

1	1	1	1	1	1
2	2	3	4	4	4
3	3	6			
4	5	7			
5					
8					

Representations of $GL_n(\mathbb{C})$

$$\{\text{polynomial irreducible representations of } GL_n(\mathbb{C})\}$$

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Weyl modules are the reps of $\mathrm{GL}_n(\mathbb{C})$ via this identification:

$$\lambda \longleftrightarrow \mathbb{S}_\lambda.$$

If $|\lambda| = m$,

- $\mathbb{S}_\lambda$ can be constructed as subspaces (or quotients) of $(\mathbb{C}^n)^{\otimes m}$
- A basis is given by the semi-standard tableaux on λ with entries $1, \ldots, n$

Weyl Modules

E.g.

$$\mathbb{S}_{(m)} = \square\square\cdots\square\square = \mathrm{Sym}^m(\mathbb{C}^n)$$

$$\mathbb{S}_{(1,\ldots,1)} = \begin{array}{c}\square\\ \square\\ \vdots\\ \square\\ \square\end{array} = \bigwedge^m(\mathbb{C}^n)$$

The Pieri Rule

Theorem (Pieri Rule)

Let μ *be a partition and* $\nu = (1, \ldots, 1)$ *be a partition of* m*. Then we have an isomorphism of* $\mathsf{GL}_n(\mathbb{C})$*-modules*

$$\mathbb{S}_\nu \otimes \mathbb{S}_\mu \cong \bigoplus_\lambda \mathbb{S}_\lambda$$

where the sum is over all $\lambda \supset \mu$ *obtained by adding* m *boxes to* μ *with no two boxes in the same row. Similarly,*

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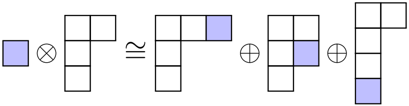
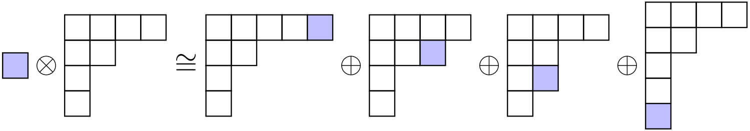
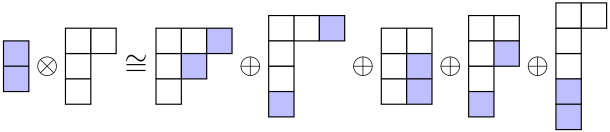
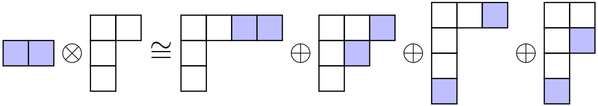




The Pieri Rule - Two Boxes





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with the general case given by iteration. We have a new, more efficient closed form description.[HMS19]

Olver's description of Pieri inclusions

$$\Phi_O = \sum_J \frac{(-1)^{|J|} J}{c_J}$$

The J are the ways to remove the indicated box out of the diagram, $|J|$ is the number of rows used, and the c_J depend on the rows used.

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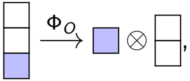

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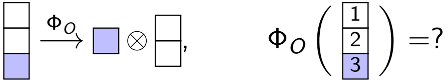

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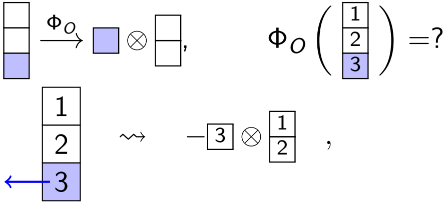

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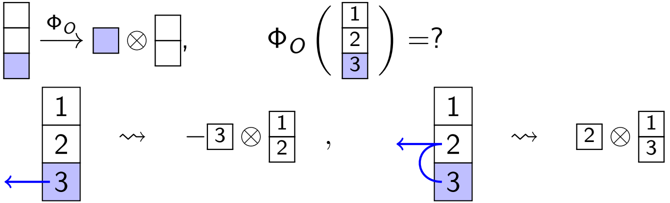

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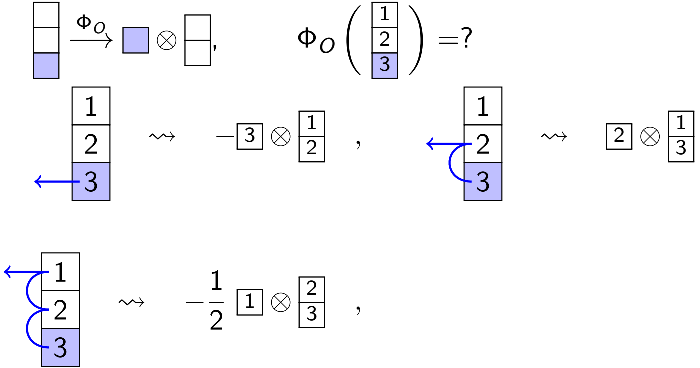

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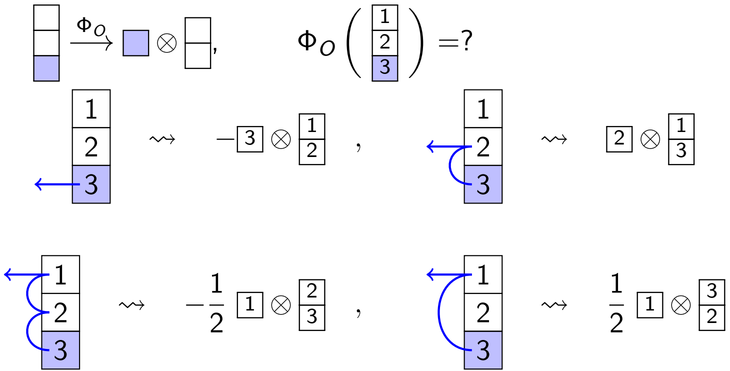

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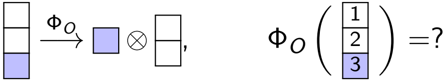

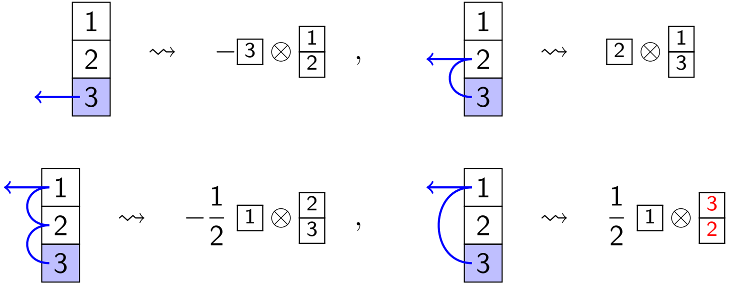

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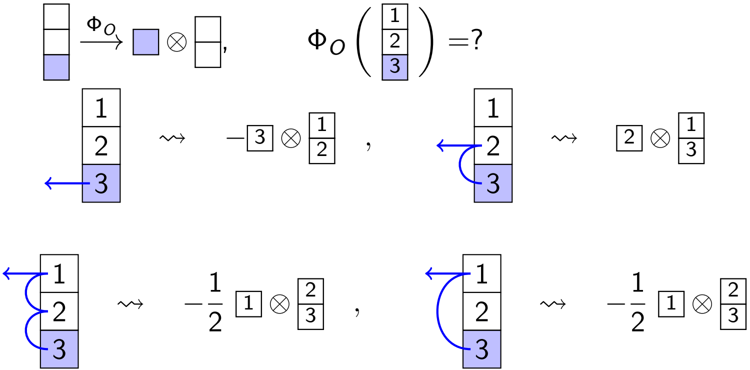

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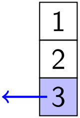

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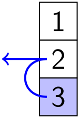
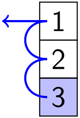


$\rightsquigarrow \quad \boxed{2} \otimes \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}$



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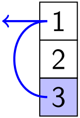

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New description of Pieri inclusions

$$\Phi = \sum_{P} \frac{(-1)^{|P|} P}{H(P)}$$

The P are the ways to remove the indicated box up and out of the diagram *with a row skipping restriction*. The coefficients $H(P)$ are similar to the c_J, but depend only on the blocks used.

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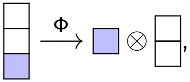
E.g., 

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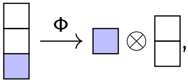
E.g., 

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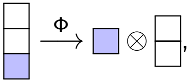
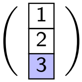
E.g., 

Φ 

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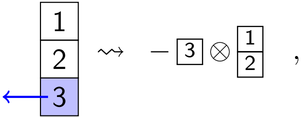

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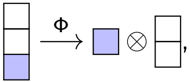

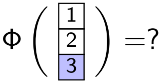

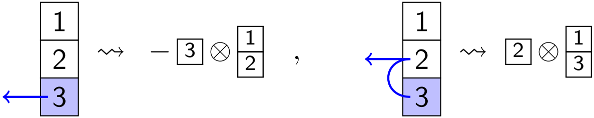

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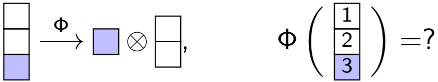
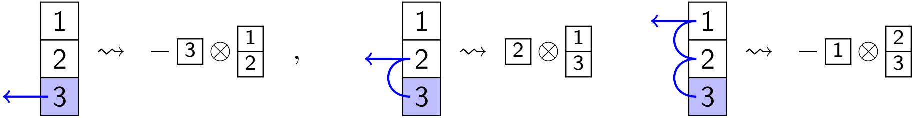
The *P* are the ways to remove the indicated box up and out of the diagram *with a row skipping restriction*. The coefficients $H(P)$ are similar to the c_J, but depend only on the blocks used.

E.g.,





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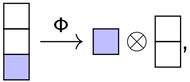
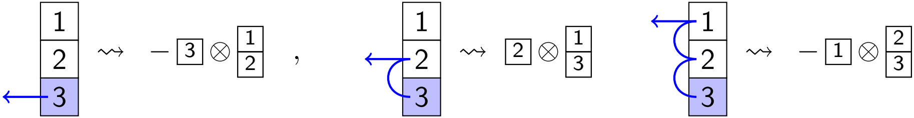
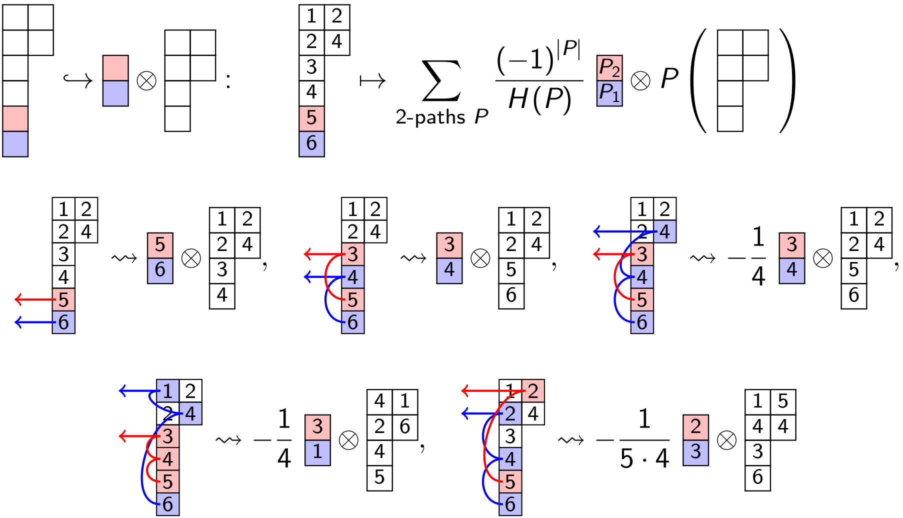


$$\Phi\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) = -\boxed{3}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \boxed{2}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} - \boxed{1}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$



Pieri inclusions removing two boxes



Computational complexity of the descriptions

Fix N and consider removing a box from the bottom row of partitions with at most N blocks.

In both descriptions, the number of terms in the Pieri inclusion acting on $\lambda = (w_1^{h_1}, \ldots, w_N^{h_N})$ depends on the number of paths on the diagram.

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New description:

$$h_1 \cdot \prod_{i=2}^{N} (h_i + 1)\,.$$

Thank You!

Markus Hunziker, John Miller, and Mark Sepanski.
Explicit pieri inclusions, 2019.
arXiv:1911.11045.