Explicit Pieri Inclusions and Minimal Graded Free Resolutions of Modules of Covariants of Several Vectors and Covectors for a General Linear Group

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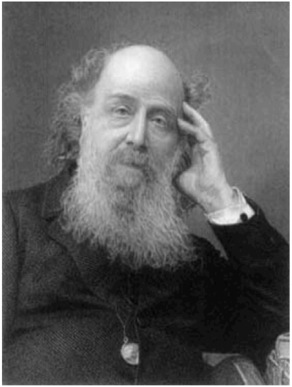
The following is joint work with Markus Hunziker and Mark Sepanski of Baylor University.

Syzygies



James Sylvester

syz·y·gy /ˈsizijē/

from the Greek $\sigma\acute{\upsilon}\zeta\upsilon\gamma o\varsigma$ [syzygos] meaning "yoked together"

Syzygies

$R = \mathbb{C}[z_1, \ldots, z_n]$ polynomial ring

$M = \langle g_1, \ldots, g_{b_0} \rangle$ fin. gen. R-module

R^{b_0} free R-module with basis $\{e_i : 1 \leq i \leq b_0\}$

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Consider the R-module map $\epsilon : R^{b_0} \to M$ given by $e_i \mapsto g_i$. Then

$$R^{b_0} \xrightarrow{\epsilon} M \longrightarrow 0 \quad \text{is exact.}$$

- An element of $\ker \epsilon$ is called a relation or **first syzygy** of M.
- $\ker \epsilon$ is called the **module of first syzygies**.

Syzygies

The module of first syzygies, $\ker \epsilon$, is also a finitely generated R-module.

Therefore, $\exists$ R-module map $\delta_1 : R^{b_1} \longrightarrow R^{b_0}$ such that

$$R^{b_1} \xrightarrow{\delta_1} R^{b_0} \xrightarrow{\epsilon} M \longrightarrow 0 \quad \text{is exact.}$$

The module of **second syzygies**, $\ker \delta_1$, is again finitely generated.

Therefore, $\exists$ R-module map $\delta_2 : R^{b_2} \longrightarrow R^{b_1}$ such that

$$R^{b_2} \xrightarrow{\delta_2} R^{b_1} \xrightarrow{\delta_1} R^{b_0} \xrightarrow{\epsilon} M \longrightarrow 0 \quad \text{is exact.}$$

etc.

Syzygies

Will this go on forever?

Syzygy Theorem, Hilbert 1890

For $R = \mathbb{C}\left[z_1, \ldots, z_n\right]$ and M as above, $\exists$ exact sequence of free modules (free resolution)

$$0 \longrightarrow R^{b_m} \xrightarrow{\delta_m} \cdots R^{b_1} \xrightarrow{\delta_1} R^{b_0} \xrightarrow{\epsilon} M \longrightarrow 0,$$

where $m \leq n$.

Classical Invariant Theory

H a group, W finite dimensional H-module

$\{x_1, \ldots, x_n\}$ coordinate functions of W

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Definition

The **coordinate ring** $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$ is the (graded) $\mathbb{C}$-algebra of polynomials from W to $\mathbb{C}$.

The grading on $\mathbb{C}[W]$ is given by

$$\mathbb{C}[W] = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathbb{C}[W]_d$$

where $\mathbb{C}[W]_d$ are the homog. poly. of degree d.

Invariants

Definition

$f \in \mathbb{C}[W]$ is called **invariant** (or H-invariant) if $f(h \cdot w) = f(w)$ for all $h \in H, w \in W$, i.e. f is constant on orbits.

Definition

The subalgebra

$$\mathbb{C}[W]^H := \{f \in \mathbb{C}[W] \mid f \text{ is } H\text{-invariant}\}$$

of $\mathbb{C}[W]$ is called the **ring of invariants**.

Covariants

Definition

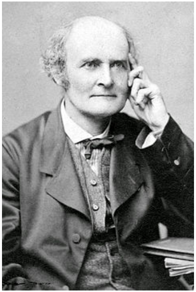
If U is another finite-dimensional H-representation, the **module of covariants of W of type U** is defined as the space

$$\mathrm{Cov}_H(W,U) := \{\phi : W \xrightarrow{\text{poly}} U \mid \phi(h \cdot w) = h \cdot \phi(w) \\ \forall\, h \in H, w \in W\}.$$

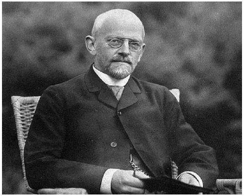
Note that if $U = \mathbb{C}$ is the trivial representation,

$$\mathrm{Cov}_H(W,U) = \mathbb{C}[W]^H.$$

Fundamental Problem of CIT



Arthur Cayley



David Hilbert

Hermann Weyl

Fundamental Problem of CIT

Find generators and syzygies (relations) for the ring of invariants $\mathbb{C}[W]^H$ and, more generally, for modules of covariants.

A partial solution to this problem for $\mathbb{C}[W]^H$ was given by Weyl in 1939.

Weyls First Fundamental Theorem

$$H = GL(V), \quad W = \underbrace{V^* \oplus \cdots \oplus V^*}_{p \text{ copies}} \oplus \underbrace{V \oplus \cdots \oplus V}_{q \text{ copies}} = (V^*)^p \oplus V^q$$

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For $1 \leq i \leq p$ and $1 \leq j \leq q$, define

$$f_{ij} : (V^*)^p \oplus V^q \to \mathbb{C}$$

by

$$f_{ij}(\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q) = \lambda_i(v_j).$$

Then, by construction, $f_{ij} \in \mathbb{C}[(V^*)^p \oplus V^q]$.

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In fact, $f_{ij} \in \mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$:

$$\begin{aligned} f_{ij}(g\cdot\ &(\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q)) \\ &= f_{ij}(g \cdot \lambda_1, \ldots, g \cdot \lambda_p, g \cdot v_1, \ldots, g \cdot v_q) \\ &= f_{ij}(\lambda_1 g^{-1}, \ldots, \lambda_p g^{-1}, g \cdot v_1, \ldots, g \cdot v_q) \\ &= \lambda_i (g^{-1} g) \cdot v_j \\ &= \lambda_i v_j = f_{ij}(\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q) \end{aligned}$$

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Theorem (FFT for $GL(V)$, Weyl 1939)

The basic invariants f_{ij} generate the invariant ring $\mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$ as a $\mathbb{C}$-algebra.

Weyls Second Fundamental Theorem

$R := \mathbb{C}\left[z_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q\right]$

$\mathbb{C}$-alg hom $\epsilon : R \to \mathbb{C}\left[(V^*)^p \oplus V^q\right]^{GL(V)}$ given by

$$z_{ij} \mapsto f_{ij}.$$

Then, by the FFT,

$$R \xrightarrow{\epsilon} \mathbb{C}\left[(V^*)^p \oplus V^q\right]^{GL(V)} \longrightarrow 0$$

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Theorem (SFT for $GL(V)$, Weyl 1939)

The first syzygies of the invariant ring $\mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$ are generated by the $(k+1) \times (k+1)$ minors of the matrix $z = (z_{ij})$, where $k = \dim V$.

Remark: The minimal graded free resolution of $\mathbb{C}\left[(V^*)^p \oplus V^q\right]^{GL(V)}$ as an R-module was first computed by Lascoux in 1978.

Main Results

Hunziker–M–Sepanski

In the context of Hermann Weyl's FT of invariant theory. for the classical groups $GL(V)$, $O(V)$, and $Sp(V)$, we are able to:

- Compute the syzygies of **all** modules of covariants **uniformly**
- Describe the differentials **explicitly**

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Idea:

- Via Howe duality: modules of covariants $\longleftrightarrow$ unitarizable highest weight modules
- These then have BGG resolutions in some parabolic category $\mathcal{O}$
- Highest weight modules $\sigma \rightsquigarrow$ "compressed" λ
- Parametrize the corresponding minimal free resolution via λ
- Parameters "unzip" to give Verma modules

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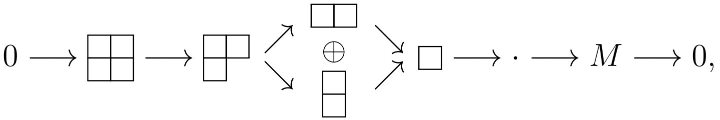
Visualization:



Today: Explain the set-up in the case $H = GL(V)$ and give an example.

Notation for $GL(V)$ Reps

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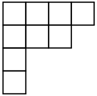
$\lambda = (\lambda_1, \ldots, \lambda_k) \quad \rightsquigarrow \quad$ Young diagram with λ_1 boxes in the first row, λ_2 boxes in the second row, etc.

We will just write the Young diagram λ for the corresponding Schur-Weyl module $\mathbb{S}_\lambda(V)$.

Notation for $GL(V)$ Reps

E.g.

$$\lambda = (4,3,1,1) \quad \rightsquigarrow \quad \mathbb{S}_\lambda(V) \longleftrightarrow \lambda =$$



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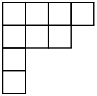

A basis for $\mathbb{S}_\lambda(V)$ is the set of **semi-standard** tableaux of shape λ.

That is, all fillings of λ with the alphabet $[k] = \{1, \ldots, k\}$ where the filling is weakly increasing across rows and strictly increasing down columns.

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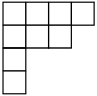
$$\lambda = (4,3,1,1) \quad \rightsquigarrow \quad \mathbb{S}_\lambda(V) \longleftrightarrow \lambda =$$ 

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E.g.

1	1	1	1
2	2	2	
3			
4			

1	2	3	3
2	3	4	
3			
4			

Notation for $GL(V)$ Reps

E.g.

$$\lambda = (4,3,1,1) \quad \rightsquigarrow \quad \mathbb{S}_\lambda(V) \longleftrightarrow \lambda = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array}$$

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E.g.

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

Note that the tableau with all ones in the first row, all twos in the second row, etc. is the highest weight vector for $\mathbb{S}_\lambda(V)$.

Notation for $GL(V)$ Reps

For general irreps of $H = GL(V)$ we have

$$\hat{H} = \{\text{irreps of } H\}$$

$$\Big\updownarrow \text{bij}$$

$$\{\sigma = (\sigma_1, \dots, \sigma_k) \mid \sigma_i \in \mathbb{Z}, \sigma_1 \geq \cdots \geq \sigma_k\}$$

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So, a general $GL(V)$ irrep is $F_\sigma^{(k)} = F_\sigma$ for some

$$\sigma = \underbrace{(n_1, \ldots, n_i, 0, \ldots, 0, -m_j, \ldots, -m_1)}_{k}$$

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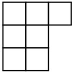
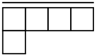
E.g. $\sigma = (3, 2, 2, 0, 0, -1, -4)$. We can associate to σ two Young diagrams:

$\sigma^+ = (3, 2, 2) =$  and $\sigma^- = (4, 1) =$ 

Back to Covariants

$$H = GL(V), \quad \dim(V) = k, \quad W = (V^*)^p \oplus V^q$$

Let $\Sigma = \left\{ \sigma \in \hat{H} \mid \mathrm{Cov}_{GL(V)}\left((V^*)^p \oplus V^q, F_\sigma\right) \neq 0 \right\}$.

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Theorem (Kashiwara-Verge 1978)

$$\sigma \in \Sigma \iff \sigma = (n_1, \ldots, n_i, 0, \ldots, 0, -m_j, \ldots, -m_1)$$

with $\quad 0 \leq i \leq q, \quad 0 \leq j \leq p$

Furthermore, for each $\sigma \in \Sigma$ we can associate a partition $\lambda(\sigma)$ to $\mathrm{Cov}_{GL(V)}\left((V^*)^p \oplus V^q, F_\sigma\right)$, namely

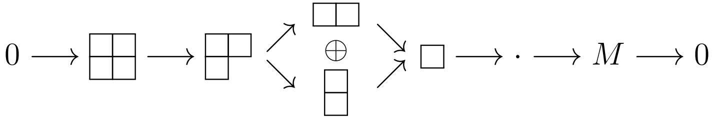
$$\lambda(\sigma) = (\underbrace{-k, \ldots, -k, -k - m_j, \ldots, -k - m_1}_{p}; \underbrace{n_1, \ldots, n_i, 0, \ldots, 0}_{q}).$$

Furthermore, the map $\sigma \mapsto \lambda(\sigma)$ is injective.

Reduction of σ

$$\sigma = (n_1, \ldots, n_i, 0, \ldots, 0, -m_j, \ldots, -m_1)$$
$$\downarrow$$
$$\lambda(\sigma) = (\underbrace{-k, \ldots, -k, -k - m_j, \ldots, -k - m_1}_{p}; \underbrace{n_1, \ldots, n_i, 0, \ldots, 0}_{q})$$
$$\downarrow$$
$$(\lambda + \rho)' = (\underbrace{\_\_, \ldots, \_\_}_{p'}; \underbrace{\_\_, \ldots, \_\_}_{q'})$$

The terms in the resolution for $M = \mathrm{Cov}_{GL(V)}\left((V^*)^p \oplus V^q, F_\sigma\right)$ will be parametrized by all Young diagrams contained in a $p' \times q'$ rectangle.



Example

$$\dim V = k = 4, \quad H = GL(V), \quad W = (V^*)^5 \oplus V^5$$

$$M = \mathrm{Cov}_{GL(V)}\left((V^*)^5 \oplus V^5, F_\sigma^{(4)}\right), \qquad p = q = 5.$$

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$$\sigma = (3, 0 - 1, -4)$$
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$$\lambda = \lambda(\sigma) = (\underbrace{-k, -k, -k, -k - m_2, -k - m_1}_{5}; \underbrace{n_1, 0, 0, 0, 0}_{5})$$

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$$\begin{gathered} \sigma = (3, 0 - 1, -4) \\ \downarrow \\ \lambda = \lambda(\sigma) = (-4, -4, -4, -5, -8; 3, 0, 0, 0, 0) \\ \downarrow \\ \lambda + \rho = (5, 4, 3, 1, -3; 7, 3, 2, 1, 0) \end{gathered}$$

Where $\rho = (p + q - 1, \ldots, 0) = (9, 8, 7, 6, 5, 4, 3, 2, 1, 0)$.

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How to get from $\lambda + \rho$ to $(\lambda + \rho)'$?

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To get $(\lambda + \rho)'$:

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To get $(\lambda + \rho)'$:

1. Delete repeats on left & right

$$\left(5, 4, \widehat{3}, \widehat{1}, -3; 7, \widehat{3}, 2, \widehat{1}, 0\right) = (5, 4, -3; 7, 2, 0)$$

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2. Delete any numbers on left $<$ all numbers on right
$$\left(5, 4, \widehat{-3}; 7, 2, 0\right) = (5, 4; 7, 2, 0)$$

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$$\left(5, 4, \widehat{-3}; 7, 2, 0\right) = (5, 4; 7, 2, 0)$$

3. Delete any numbers on right $>$ all numbers on left

$$\left(5, 4; \widehat{7}, 2, 0\right) = (5, 4; 2, 0)$$

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So $(\lambda + \rho)' = (5, 4; 2, 0)$, and $p' = q' = 2$.

Example

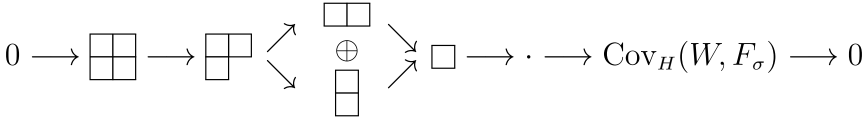
$$H = \mathrm{GL}\left(V\right), \quad k = 4, \quad W = \left(V^*\right)^5 \oplus V^5, \quad \sigma = (3, 0, -1, -4)$$

$$\rightsquigarrow (\lambda + \rho)' = (5, 4; 2, 0), \quad p' = q' = 2$$

Example

$$H = \mathrm{GL}(V), \quad k = 4, \quad W = (V^*)^5 \oplus V^5, \quad \sigma = (3, 0, -1, -4)$$

$$\rightsquigarrow (\lambda + \rho)' = (5, 4; 2, 0), \quad p' = q' = 2$$



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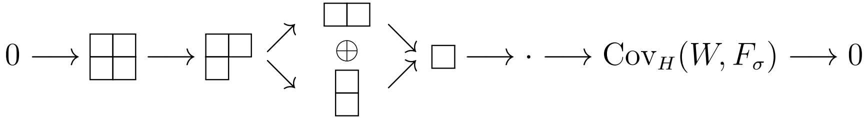

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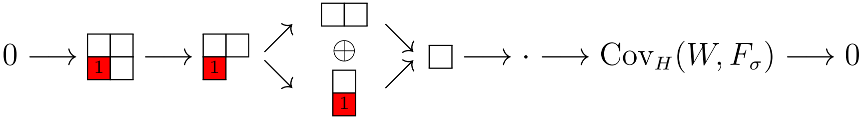

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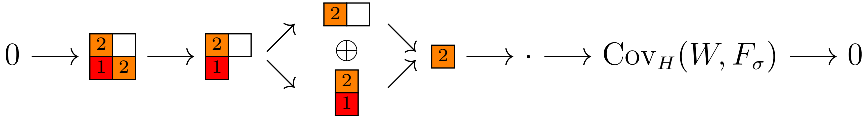

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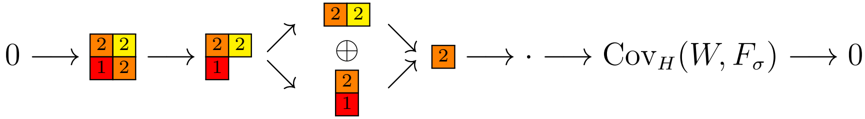

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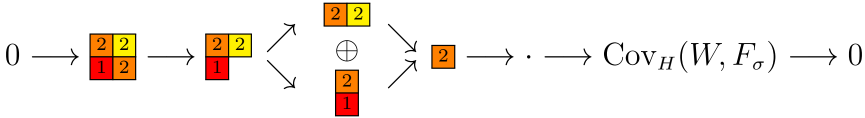

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This “compressed” version needs to be unzipped.

Example

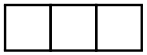
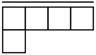
Unzip

2	2
1	

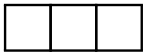
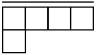
:

Example

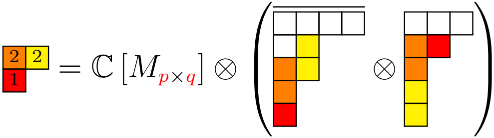
Unzip :

$\sigma = (3, 0, -1, -4) \rightsquigarrow \quad \sigma^+ = (3) =$  $\quad \sigma^- = (4, 1) =$ 

Unzip :

$\sigma = (3, 0, -1, -4) \rightsquigarrow$ $\sigma^+ = (3) =$  $\sigma^- = (4, 1) =$ 

Add boxes to the columns of σ^- and σ^+ as prescribed by the colored diagram and its transpose, respectively.



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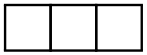
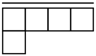
$\sigma = (3, 0, -1, -4) \rightsquigarrow \quad \sigma^+ = (3) =$  $\quad \sigma^- = (4, 1) =$ 

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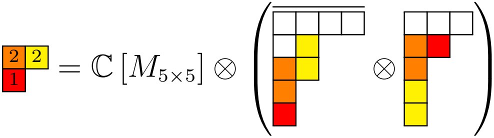
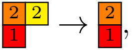


Then the maps in the resolution look like



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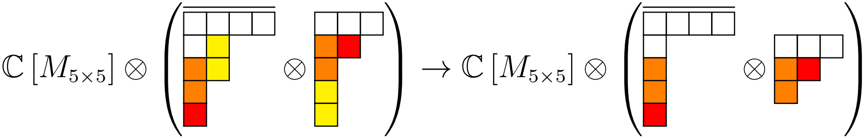

Here, each tableau on the left is losing **2** boxes.

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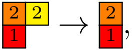

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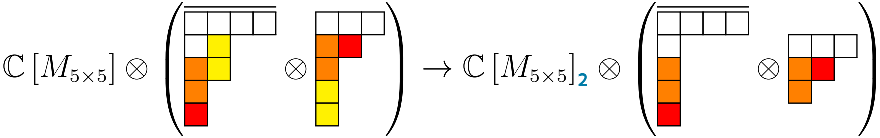

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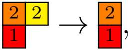

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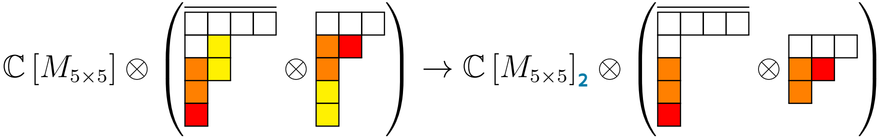
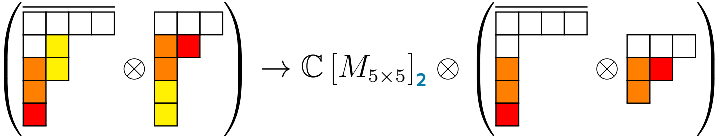
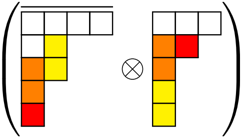


Here, each tableau on the left is losing **2** boxes.

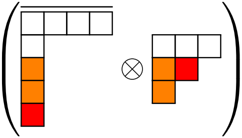
As a $\mathbb{C}[M_{5\times 5}]$-module map, it is enough to consider



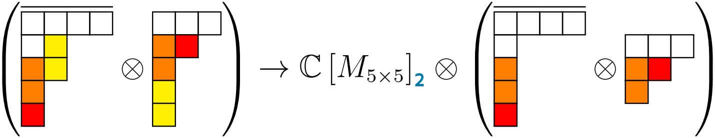
Example



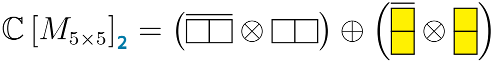
$$\to \mathbb{C}\left[M_{5\times 5}\right]_{2} \otimes$$



Example

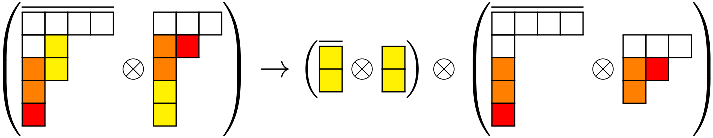


As a $\mathrm{GL}(5;\mathbb{C}) \times \mathrm{GL}(5;\mathbb{C})$ rep, $\mathbb{C}[M_{5\times5}]_{\mathbf{2}}$ decomposes into



In general, $\mathbb{C}[M_{5\times5}]_{\mathbf{d}}$ decomposes as above with all shapes that have **d** boxes and ≤ 5 rows.

Example



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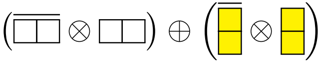

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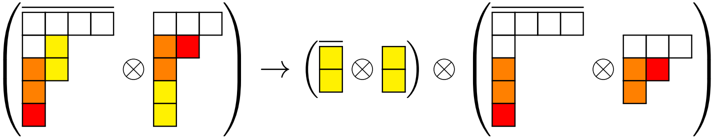
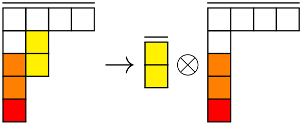
Then, we end up with...

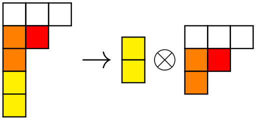
Example



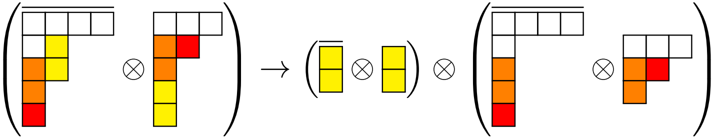
Computing the syzygies for $\mathrm{Cov}_{GL(V)}\left((V^*)^5 \oplus V^5, F_\sigma^{(4)}\right)$ then comes down to finding $\mathbb{C}[M_{5\times 5}]$-module and $\mathrm{GL}(V)$-equivariant **Pieri inclusions**, such as



and



Example



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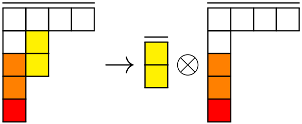

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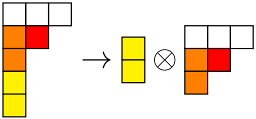


For the rest of this talk: updated description of Pieri inclusions.

The Pieri Rule

Theorem (Pieri Rule)

Let μ be a partition corresponding to a Schur-Weyl module $\mathbb{S}_{\mu}(V)$ and $\nu = (1, \ldots, 1)$ be a partition of m. Then we have an isomorphism of $\mathrm{GL}(V)$-modules

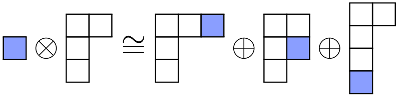
$$\mathbb{S}_{\nu}(V) \otimes \mathbb{S}_{\mu}(V) \cong \bigoplus_{\lambda} \mathbb{S}_{\lambda}(V)$$

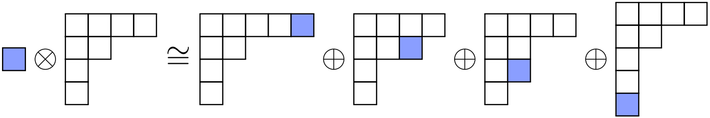
where the sum is over all $\lambda \supset \mu$ obtained by adding m boxes to μ with no two boxes in the same row. Similarly,

$$\mathbb{S}_{(m)}(V) \otimes \mathbb{S}_{\mu}(V) \cong \bigoplus_{\lambda} \mathbb{S}_{\lambda}(V)$$

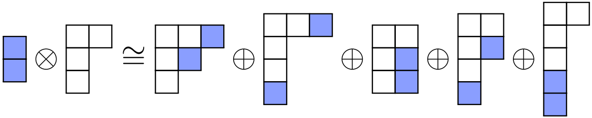
where the sum is over all $\lambda \supset \mu$ obtained by adding m boxes to μ with no two boxes in the same column.

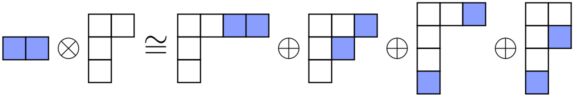
The Pieri Rule - One Box



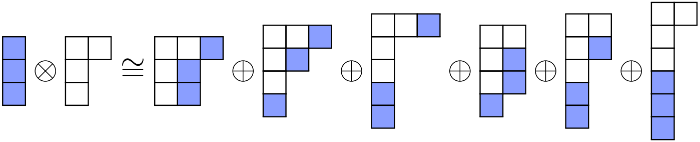


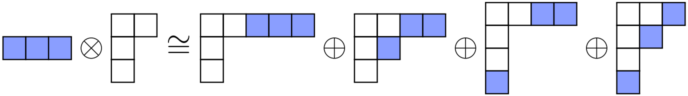
The Pieri Rule - Two Boxes





The Pieri Rule - Three Boxes





Pieri Inclusions

From the Pieri rule we get maps

$$\mathbb{S}_\lambda(V) \to \mathbb{S}_\nu(V) \otimes \mathbb{S}_\mu(V),$$

called **Pieri inclusions**, unique up to non-zero scalar multiple.

E.g.

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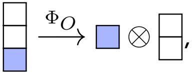
These Pieri inclusions were first given by Olver in his thesis (1982) via iterating one box removal, and made more explicit by Sam (2009) and Sam and Weyman (2012).

We give a general closed form description of Pieri inclusions removing m boxes and show this rule is more efficient.

Olver's description of Pieri inclusions

$$\Phi_O = \sum_J \frac{(-1)^{|J|} J}{c_J}$$

The J are the ways to remove the indicated box up and out of the diagram, $|J|$ is the number of rows used, and the c_J depend on the rows used.

E.g., 

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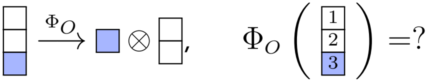

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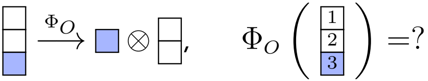

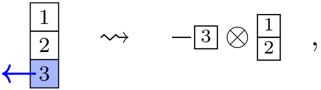

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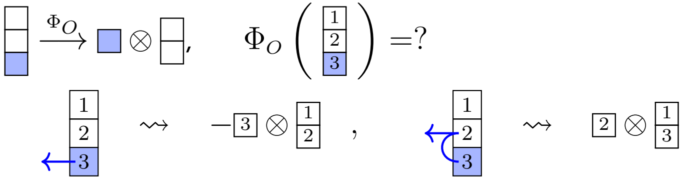

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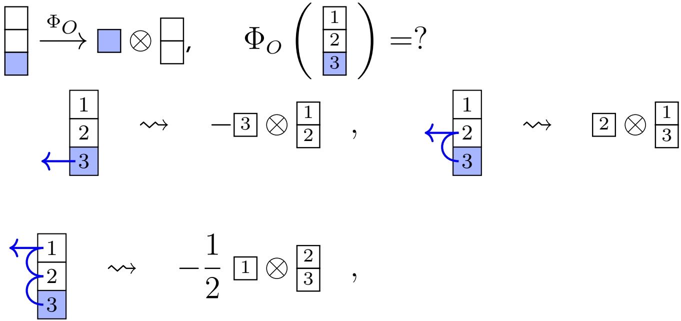

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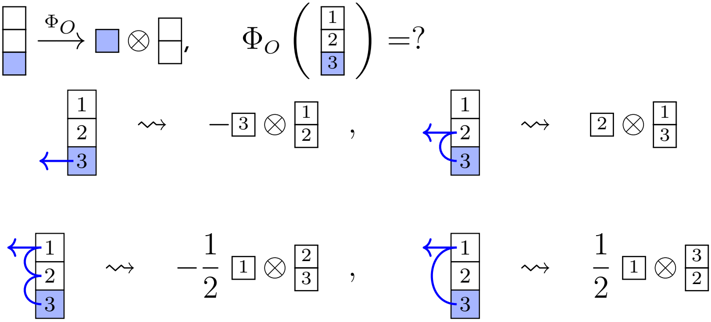

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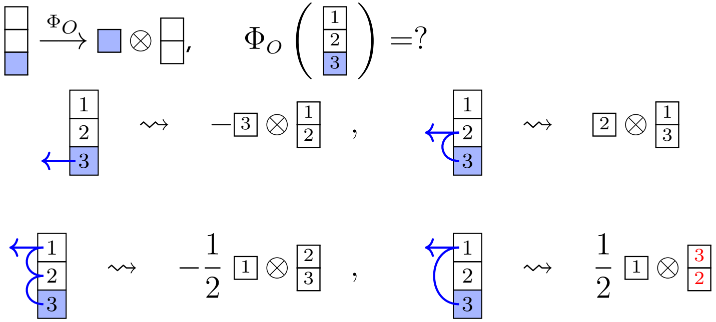

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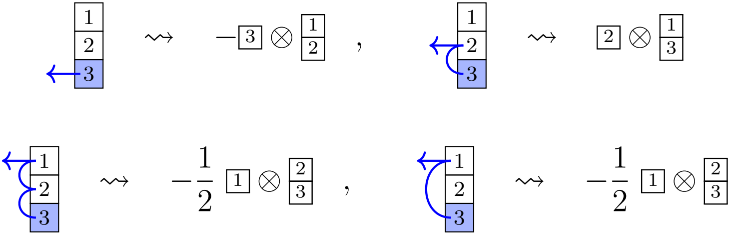

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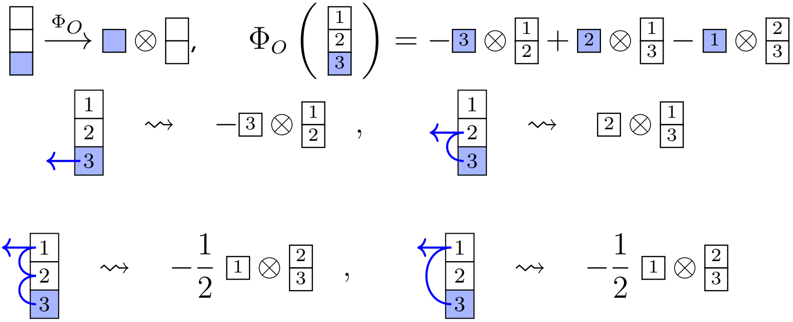

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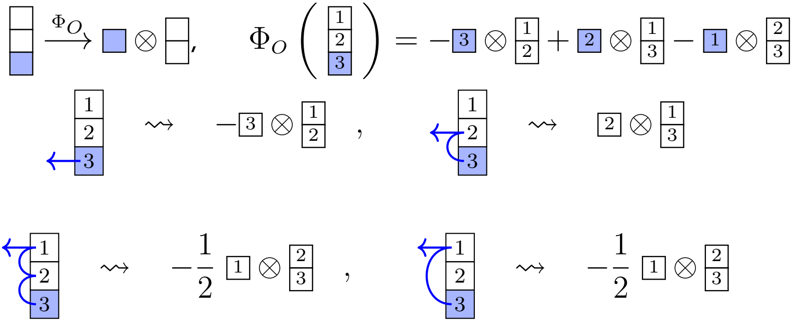

Blocks of a diagram

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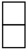

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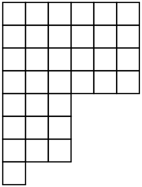

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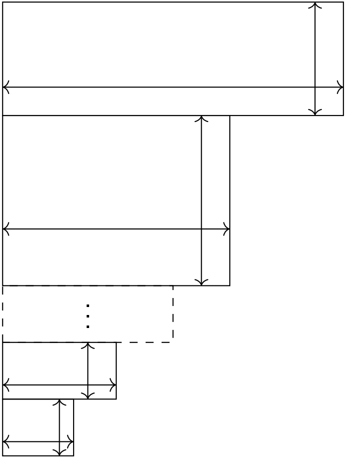

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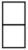

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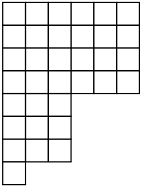

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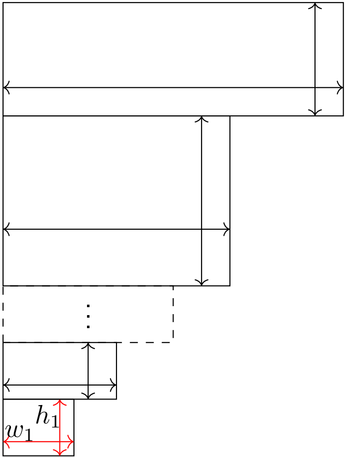

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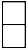

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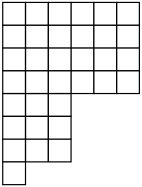

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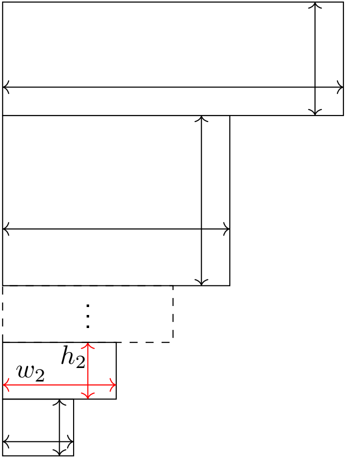

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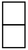

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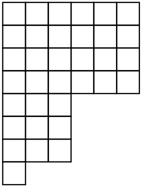

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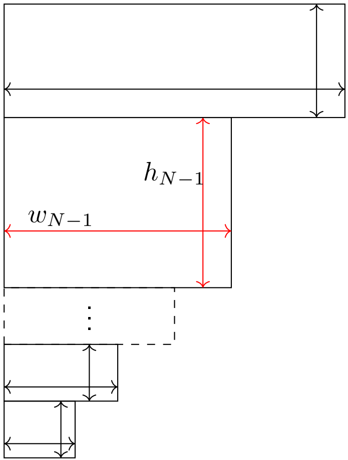

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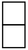

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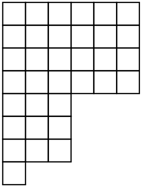

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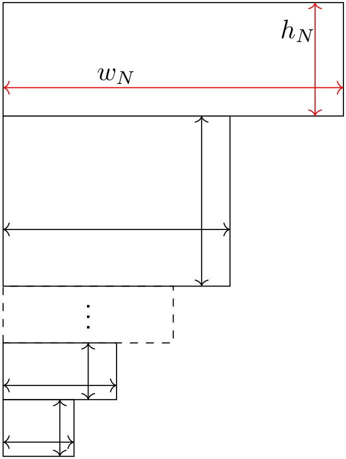

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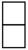

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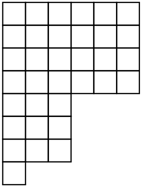
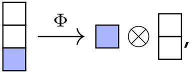


New description of Pieri inclusions

$$\Phi = \sum_{P} \frac{(-1)^{|P|} P}{H(P)}$$

The P are the ways to remove the indicated box up and out of the diagram **with a row skipping restriction**. The coefficients $H(P)$ are similar to the c_J, but **depend only on the blocks used**.

E.g., 

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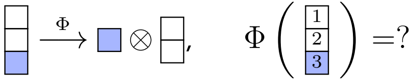

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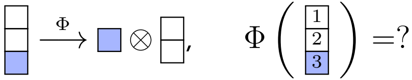

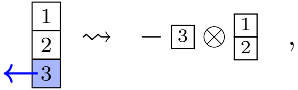

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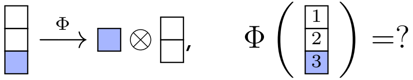

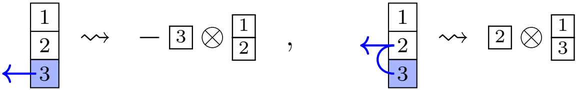

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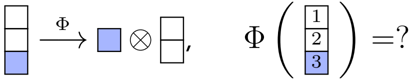
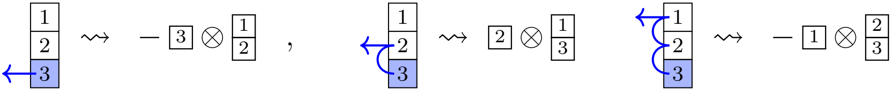
E.g., 



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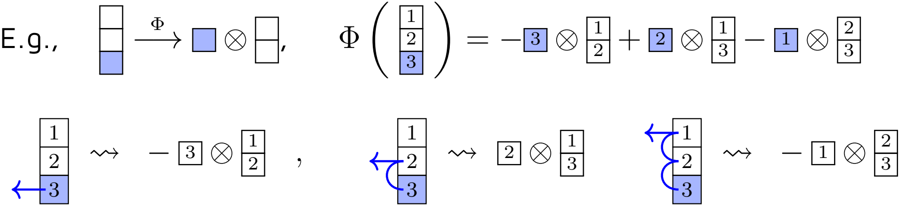
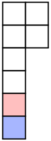
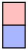
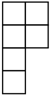
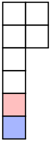
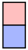
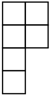


Pieri inclusions removing two boxes

 $\hookrightarrow$  $\otimes$  :

 $\mapsto \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)} P_{out}\Big($  $\Big) \otimes P \Big($  $\Big)$

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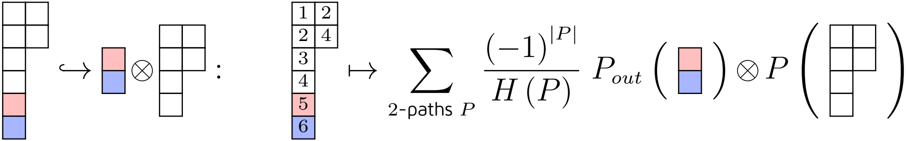

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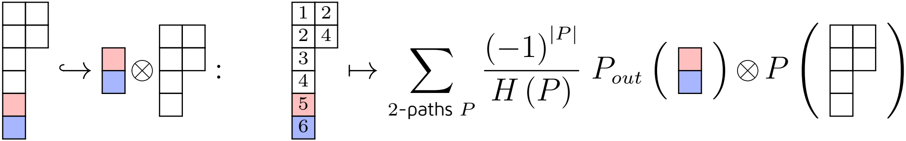

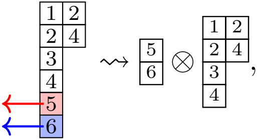

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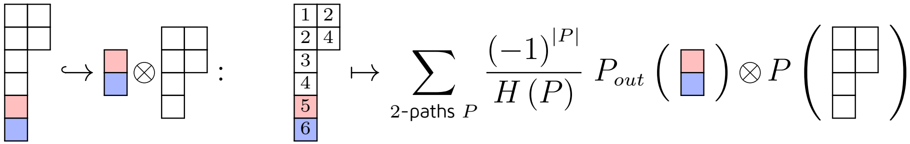

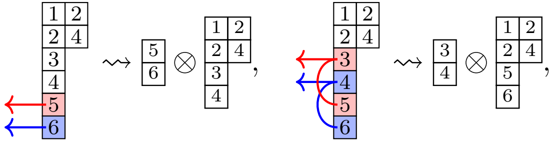

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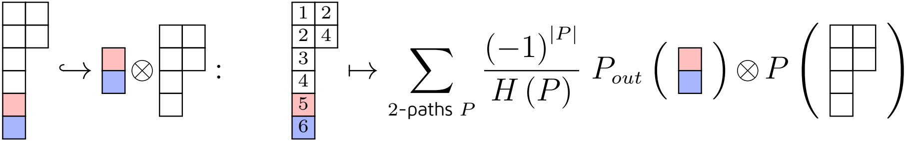

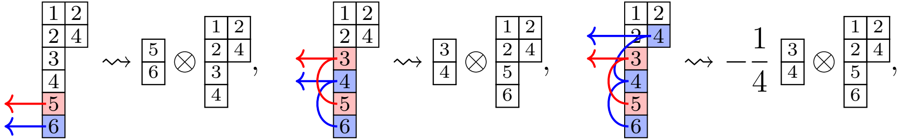

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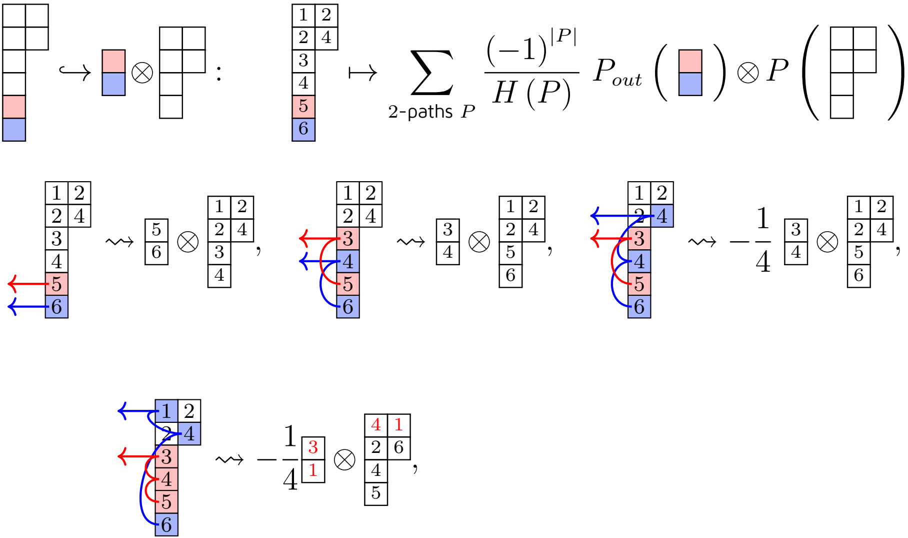

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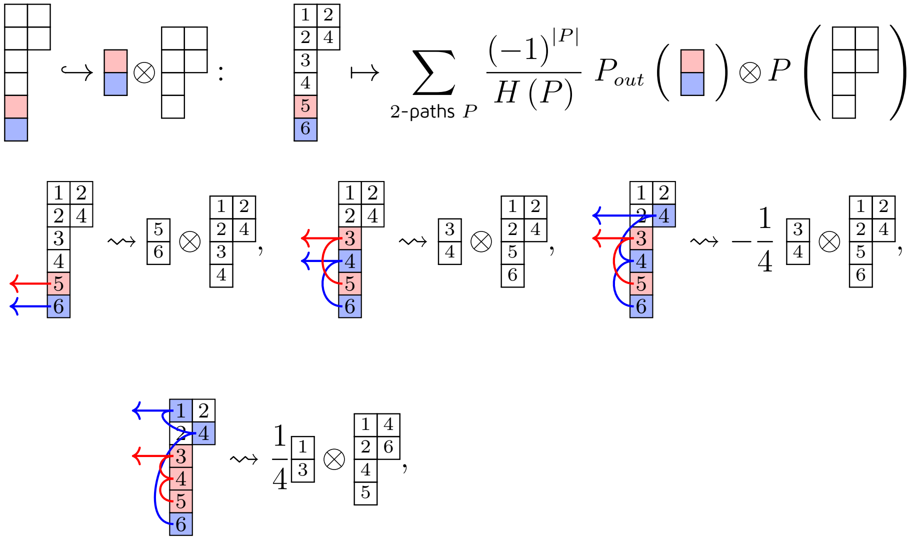

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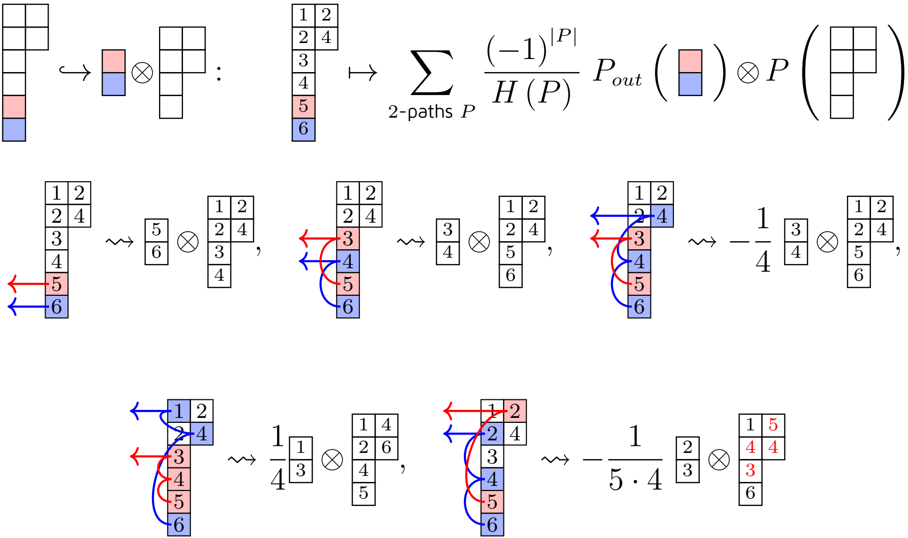

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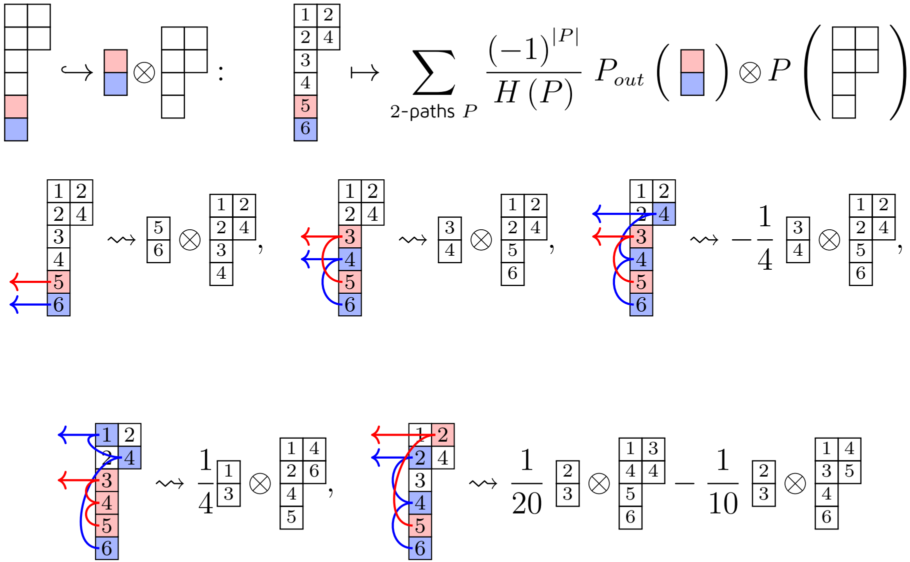

Complexity of the descriptions

In both the old and new descriptions, the number of terms in the Pieri inclusion acting on $\lambda = (w_1^{h_1}, \ldots, w_N^{h_N})$ depends on the number of paths acting on the diagram.

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New description: paths given by the number of choices of rows in the diagram, where you must choose the first row **and cannot skip rows within blocks**.

$$h_1 \cdot \prod_{i=2}^{N} (h_i + 1) .$$

Computation Time in Macaulay2

Old description:

```
i3 : time pieri({6,6,6}, {3}, CC[a,b,c])
     -- used 0.723831 seconds

o3 = | 6c   |
     | -36b |
     | 216a |
```

New description:

```
i31 : time pieri({6,6,6}, {3}, CC[a,b,c])
     -- used 0.0219184 seconds

o31 = | 6c   |
      | -36b |
      | 216a |
```

Computation Time in Macaulay2

Old description:

```
i4 : time pieri({7,7,7}, {3}, CC[a,b,c])
     -- used 88.0308 seconds

o4 = | 7c   |
     | -49b |
     | 343a |
```

New description:

```
i32 : time pieri({7,7,7}, {3}, CC[a,b,c])
     -- used 0.0350425 seconds

o32 = | 7c   |
      | -49b |
      | 343a |
```

Computation Time in Macaulay2

Old description:

```
i3 : time pieri({8,8,8}, {3}, CC[a,b,c])
  ^C ^Cstdio:3:6:(3): error: interrupted
     -- used 3538.41 seconds
```

New description:

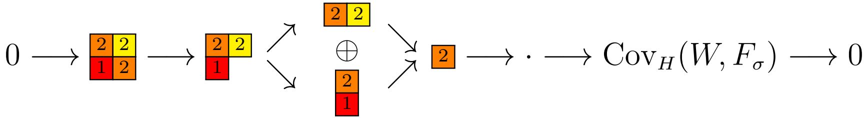
```
i33 : time pieri({8,8,8}, {3}, CC[a,b,c])
     -- used 0.0688948 seconds

o33 = | 8c   |
      | -64b |
      | 512a |
```

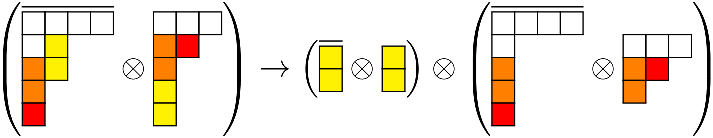
Back to the syzygies example

Recall:

$$W = (V^*)^5 \oplus V^5, \quad \sigma = (3, 0, -1, -4)$$



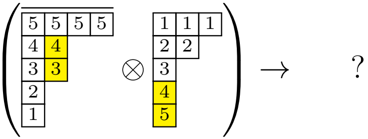
And after unzipping, the maps look like



and are induced by Pieri inclusions.

Back to the syzygies example

Consider the image of the highest weight vector:



We will apply the Pieri inclusion map removing two boxes to each term separately, then put them together.

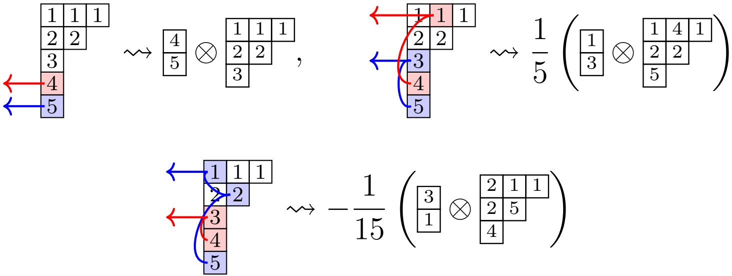
Back to the syzygies example

$$\Phi\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}\right) = \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)} P_{out}\left(\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right) \otimes P\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}\right)$$

Back to the syzygies example

$$\Phi\left(\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&\multicolumn{1}{|c}{}\\\cline{1-2} 3&\multicolumn{2}{|c}{}\\\cline{1-1} 4&\multicolumn{2}{|c}{}\\\cline{1-1} 5&\multicolumn{2}{|c}{}\\\cline{1-1}\end{array}\right) = \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)}\, P_{out}\left(\begin{array}{|c|}\hline \\\hline \\\hline\end{array}\right) \otimes P\left(\begin{array}{|c|c|c|}\hline &&\\\hline &&\multicolumn{1}{|c}{}\\\cline{1-2} &\multicolumn{2}{|c}{}\\\cline{1-1}\end{array}\right)$$

Some terms in the sum are (pre-straightening):



Back to the syzygies example

$$
\begin{aligned}
\Phi\left(\ytableaushort{111,22,3,4,5}\right) = {} & \ytableaushort{4,5}\otimes\ytableaushort{111,22,3} - \ytableaushort{3,5}\otimes\ytableaushort{111,22,4} - \frac{2}{3}\left(\ytableaushort{2,5}\otimes\ytableaushort{111,24,3}\right) \\
& \frac{2}{3}\left(\ytableaushort{2,5}\otimes\ytableaushort{111,23,4}\right) - \frac{3}{5}\left(\ytableaushort{1,5}\otimes\ytableaushort{114,22,3}\right) \\
& - \frac{2}{5}\left(\ytableaushort{1,5}\otimes\ytableaushort{112,23,4}\right) + \frac{2}{5}\left(\ytableaushort{1,5}\otimes\ytableaushort{112,24,3}\right) \\
& + \frac{3}{5}\left(\ytableaushort{1,5}\otimes\ytableaushort{113,22,4}\right) + \ytableaushort{3,4}\otimes\ytableaushort{111,22,5} \\
& + \cdots
\end{aligned}
$$

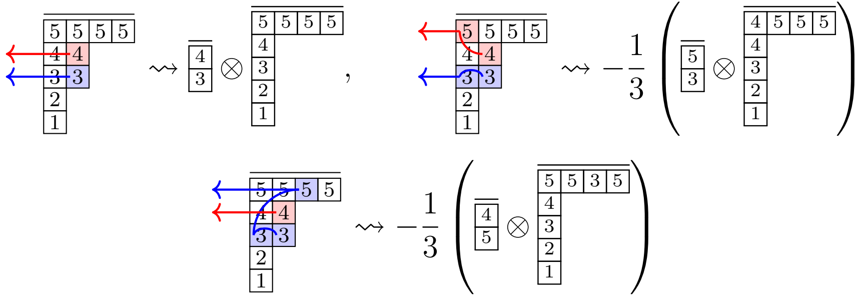
Back to the syzygies example

$$\Phi\left(\overline{\begin{array}{|c|c|c|c|}\hline 5 & 5 & 5 & 5 \\ \hline 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}}\right) = \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)}\, P_{out}\left(\overline{\begin{array}{|c|}\hline \square \\ \hline \square \\ \hline \end{array}}\right) \otimes P\left(\overline{\begin{array}{|c|c|c|c|}\hline \square & \square & \square & \square \\ \hline \square & \multicolumn{3}{c}{} \\ \cline{1-1} \square & \multicolumn{3}{c}{} \\ \cline{1-1} \square & \multicolumn{3}{c}{} \\ \cline{1-1} \square & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}}\right)$$

Back to the syzygies example

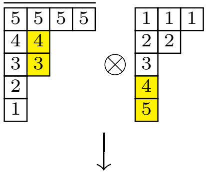
$$\Phi\left(\overline{\begin{array}{|c|c|c|c|}\hline 5&5&5&5\\\hline 4&4\\\cline{1-2} 3&3\\\cline{1-2} 2\\\cline{1-1} 1\\\cline{1-1}\end{array}}\right) = \sum_{\text{2-paths } P} \frac{(-1)^{|P|}}{H(P)}\, P_{out}\left(\overline{\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}}\right) \otimes P\left(\overline{\begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline \ \\\cline{1-1} \ \\\cline{1-1} \ \\\cline{1-1} \ \\\cline{1-1}\end{array}}\right)$$

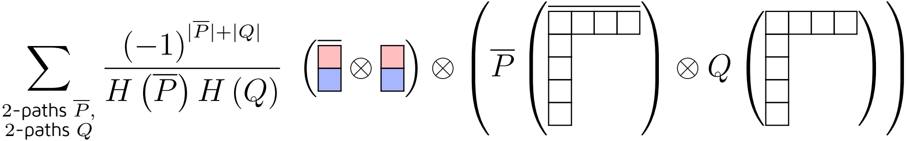
Some terms in the sum are (pre-straightening):



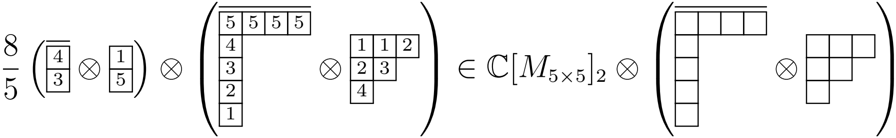
$$\Phi\left(\overline{\begin{ytableau} 5 & 5 & 5 & 5 \\ 4 & *(red!30) 4 \\ 3 & *(blue!30) 3 \\ 2 \\ 1 \end{ytableau}}\right) = 4\left(\overline{\begin{ytableau} 4 \\ 3 \end{ytableau}} \otimes \overline{\begin{ytableau} 5 & 5 & 5 & 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{ytableau}}\right) - \frac{16}{3}\left(\overline{\begin{ytableau} 5 \\ 3 \end{ytableau}} \otimes \overline{\begin{ytableau} 5 & 5 & 5 & 4 \\ 4 \\ 3 \\ 2 \\ 1 \end{ytableau}}\right)$$

$$+ \frac{16}{3}\left(\overline{\begin{ytableau} 5 \\ 4 \end{ytableau}} \otimes \overline{\begin{ytableau} 5 & 5 & 5 & 3 \\ 4 \\ 3 \\ 2 \\ 1 \end{ytableau}}\right)$$





E.g. one term in the image is



Back to the syzygies example

Still need $\overline{\begin{array}{|c|}\hline 4\\\hline 3\\\hline\end{array}} \otimes \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} \in \mathbb{C}[M_{5\times 5}]_2$, given by a minor:

$$\begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix}$$

Back to the syzygies example

Still need $\overline{\begin{array}{|c|}\hline 4\\\hline 3\\\hline\end{array}} \otimes \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} \in \mathbb{C}[M_{5\times 5}]_2$, given by a minor:

$$\begin{array}{c} \\ \\ \overline{\boxed{3}} \\ \overline{\boxed{4}} \\ \\ \end{array} \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix}$$

Back to the syzygies example

Still need $\overline{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline\end{array}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline\end{array} \in \mathbb{C}[M_{5\times 5}]_2$, given by a minor:

$$\begin{array}{cc} & \begin{array}{ccccc} \boxed{1} & & & & \boxed{5} \end{array} \\ \begin{array}{c} \\ \\ \overline{\boxed{3}} \\ \overline{\boxed{4}} \\ \\ \end{array} & \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \end{array}$$

Back to the syzygies example

Still need $\overline{\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline\end{array}} \otimes \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline\end{array} \in \mathbb{C}[M_{5\times 5}]_2$, given by a minor:

$$\begin{array}{cc} & \begin{array}{ccccc} \boxed{1} & & & & \boxed{5} \end{array} \\ \begin{array}{c} \\ \\ \overline{\boxed{3}} \\ \overline{\boxed{4}} \\ \\ \end{array} & \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \end{array}$$

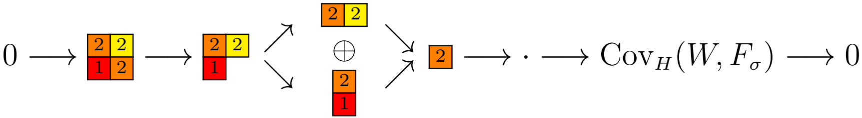
Back to the syzygies example

Still need $\overline{\begin{array}{|c|}\hline 4\\\hline 3\\\hline\end{array}} \otimes \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} \in \mathbb{C}[M_{5\times 5}]_2$, given by a minor:

$$\begin{array}{cc} & \begin{array}{ccccc} \boxed{1} & & & & \boxed{5} \end{array} \\ \begin{array}{c} \\ \\ \overline{\boxed{3}} \\ \overline{\boxed{4}} \\ \\ \end{array} & \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \end{array}$$

$$\overline{\begin{array}{|c|}\hline 4\\\hline 3\\\hline\end{array}} \otimes \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array} = z_{31}z_{45} - z_{35}z_{41}.$$

- "Unzip" each term.
- Compute the differentials similarly.
- Generally, for any $W = (V^*)^p \oplus V^q$ and any σ.



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