

# How to Multiply Young Tableaux via the Teasing Game

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Spring 2021 MAA Missouri Section Meeting

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Slides available: [johnamiller.xyz](http://johnamiller.xyz)

## The fifteen puzzle



*https:*  
*//lorecioni.github.io/fifteen-puzzle-game/*

## What is a Young tableau?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

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An array of adjacent boxes so that

- the array is upper-left justified
- from top to bottom the number of boxes in the rows does not increase
- each box is filled with a positive integer so that
  - across rows: non-decreasing
  - down columns: increasing

## Which one of these is a Young tableau?

(A).

2	2	3	4		
3	4	7	8	9	9
5	5	8			

(D).

1	1	1	2	1
2	2	3	3	
3	4	5		

(B).

1	2	3		
			4	
2	3	5		
3				

(E).

1	3	3
	4	

(C).

3	3	4	4	5
3	7	9		
4				

(F).

4	5	6	6
5	6		
7	8		
9			

## Come up with your own Young tableau

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Create your own Young tableau! Can you make one that:

- has exactly seven boxes
- or has no repeating numbers,
- or has only one row/column?

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2	2	2
3	6	
7	9	

1
2
3
4
5
6
7



# Young tableaux are powerful!

## The Calculus of Tableaux

- Certain types of YT  $\leftrightarrow$  Words and the plactic monoid
- Count certain types of YT  $\rightsquigarrow$  Littlewood-Richardson Rule for decomposing products of Schur functions

## Representation Theory

- Certain types of YT parametrize all irreducible representations of  $\mathfrak{S}_n$  (similarly  $GL(V)$ ), count options  $\rightsquigarrow$  dimension

**and more!**

**The point:** knowing about how YT work  $\rightsquigarrow$  knowing how “harder” things work

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This multiplication is defined by playing the jeu de taquin (“teasing game”) with two Young tableaux, which involves sliding moves just as in the fifteen puzzle!

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To find  $\begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & & \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} :$

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2			



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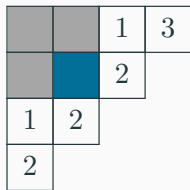
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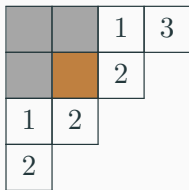
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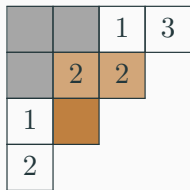
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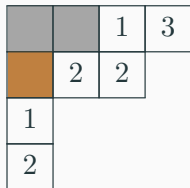
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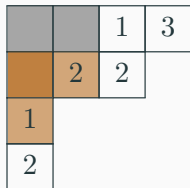


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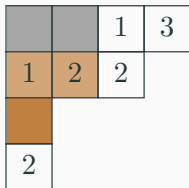
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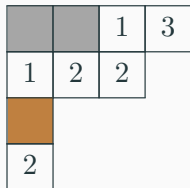
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## Jeu de taquin (teasing game)

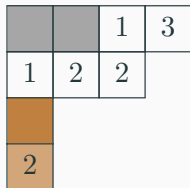
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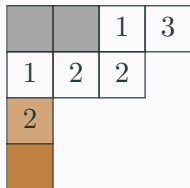
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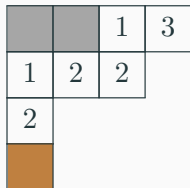
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		1	3
1	2	2	
2			

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	1	2	3
1	2		
2			

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1	2		
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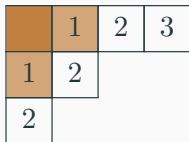
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1	2		
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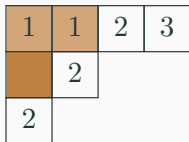
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1	1	2	3
	2		
2			

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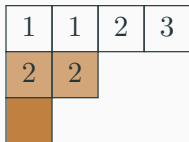
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2	2		

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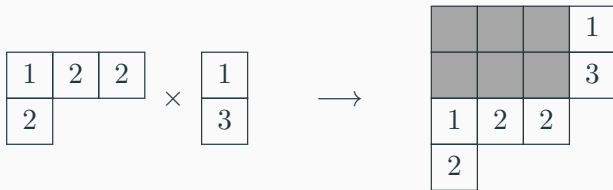
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2	2		

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## The product on YT via the jeu de taquin



jeu de taquin



1	1	2	3
2	2		

Why is  $\times$  “nice”?

- If  $T$  and  $S$  are two Young tableaux,  $T \times S$  is also a Young tableau.
  - Forced by choice of which boxes to slide.
- $\times$  is associative.
  - Since order of bottom right gray boxes doesn't matter.
- The empty diagram (with no boxes) is the unit.
  - If using empty diagram then there are no gray boxes from the start.





## Non-commutativity

While  $\times$  is nice enough, it is not commutative:

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline \text{ } & \text{ } & 1 \\ \hline 1 & 2 & \text{ } \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline \text{ } & 1 & 2 \\ \hline 1 & \text{ } & \text{ } \\ \hline \end{array}$$

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$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

		1
1	2	

	1	2
1		

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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

		1
1	2	

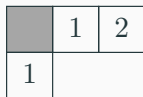
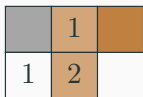
	1	2
1		

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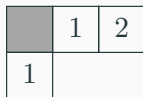
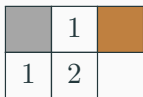


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$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$



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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

	1	
1	2	

	1	2
1		

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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

	1
1	2

	1	2
1		



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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

	1
1	2

	1	2
1		

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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

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1	1
1	2

	1	2
1		

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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

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1	1
	2

	1	2
1		

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1	1
	2

	1	2
1		

## Non-commutativity

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$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$$

1	1
2	2

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1		

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$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|} \hline \text{brown} & 1 & 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array}$$

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$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline \color{red} & & \\ \hline \end{array}$$

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