Pieri Inclusions and Syzygies of Modules of Covariants of Several Vectors and Covectors

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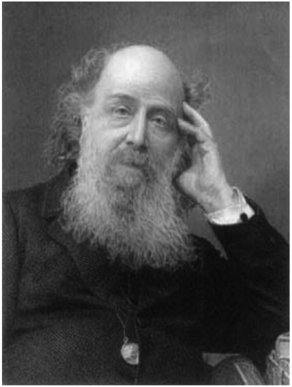
Syzygies



James Sylvester

syz·y·gy /'sizijē/

from the Greek $\sigma\acute{\upsilon}\zeta\upsilon\gamma o\varsigma$ [syzygos] meaning "yoked together"

Syzygies

$S = \mathbb{C}[z_1, \ldots, z_n]$ polynomial ring

$M = \langle g_1, \ldots, g_r \rangle$ finitely generated (graded) S-module

S^r free S-module with basis $\{e_i : 1 \leq i \leq r\}$

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S^r free S-module with basis $\{e_i : 1 \leq i \leq r\}$

Consider the S-module map $\epsilon : S^r \to M$ given by $e_i \mapsto g_i$. Then

$$S^r \xrightarrow{\epsilon} M \longrightarrow 0 \quad \text{is exact.}$$

An element of $\ker \epsilon$ is called a relation or *first syzygy* of M, and $\ker \epsilon$ is called the *module of first syzygies*.

Syzygies

The module of first syzygies, $\ker \epsilon$, is also a finitely generated S-module, so $\exists$ S-module map $\delta_1 : S^{r_1} \longrightarrow S^r$ such that

$$S^{r_1} \xrightarrow{\delta_1} S^r \xrightarrow{\epsilon} M \longrightarrow 0 \quad \text{is exact.}$$

The module of *second syzygies*, $\ker \delta_1$, is again finitely generated, so $\exists$ S-module map $\delta_2 : S^{r_2} \longrightarrow S^{r_1}$ such that

$$S^{r_2} \xrightarrow{\delta_2} S^{r_1} \xrightarrow{\delta_1} S^r \xrightarrow{\epsilon} M \longrightarrow 0 \quad \text{is exact.}$$

etc.

Syzygies

Will this go on forever?

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Theorem (Syzygy Theorem, Hilbert 1890)

For $S = \mathbb{C}[z_1, \ldots, z_n]$ and M as above, $\exists$ exact sequence of free modules (free resolution)

$$0 \longrightarrow S^{r_m} \xrightarrow{\delta_m} \cdots \longrightarrow S^{r_2} \xrightarrow{\delta_2} S^{r_1} \xrightarrow{\delta_1} S^{r} \xrightarrow{\epsilon} M \longrightarrow 0,$$

where $m \leq n$.

Classical Invariant Theory

Syzygies were first studied in the context of **classical invariant theory**

H a group, W finite dimensional H-module

$\{x_1, \ldots, x_n\}$ coordinate functions of W

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Definition

The **coordinate ring** $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$ is the (graded) $\mathbb{C}$-algebra of polynomials from W to $\mathbb{C}$.

The grading on $\mathbb{C}[W]$ is given by

$$\mathbb{C}[W] = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathbb{C}[W]_d$$

where $\mathbb{C}[W]_d$ is the space of homogeneous polynomials of degree d.

Invariants

Definition

$f \in \mathbb{C}[W]$ is called **invariant** (or H-invariant) if $f(h \cdot w) = f(w)$ for all $h \in H, w \in W$, i.e. f is constant on orbits.

Note that f is invariant $\iff h \cdot f = f \quad$ for all $h \in H$, i.e. f is fixed by the regular action of H on $\mathbb{C}[W]$.

Definition

The subalgebra $\mathbb{C}[W]^H := \{f \in \mathbb{C}[W] \mid f \text{ is invariant}\}$ of $\mathbb{C}[W]$ is called the **ring of invariants**.

Covariants

Definition

If U is another finite-dimensional H-representation, the **module of covariants of W of type U** is defined as the space

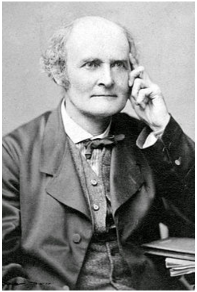
$$\mathrm{Cov}_H(W,U) := \left\{ \phi : W \xrightarrow{\text{poly}} U \mid \phi(h\cdot w) = h\cdot\phi(w) \quad \forall\, h\in H, w\in W \right\}$$

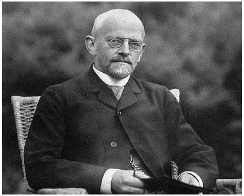
with the obvious $\mathbb{C}[W]^H$-module structure.

Note that if $U = \mathbb{C}$ is the trivial representation,

$$\mathrm{Cov}_H(W,U) = \mathbb{C}[W]^H.$$

Fundamental Problem of CIT


Arthur Cayley


David Hilbert


Hermann Weyl

Fundamental Problem of CIT

Find generators and relations (syzygies) for the ring of invariants $\mathbb{C}[W]^H$ *and, more generally, for modules of covariants.*

A partial solution to this problem for $\mathbb{C}[W]^H$ was given by Weyl in 1939.

Weyls First Fundamental Theorem

$H = GL(V)$

$$W = \underbrace{V^* \oplus \cdots \oplus V^*}_{p \text{ copies}} \oplus \underbrace{V \oplus \cdots \oplus V}_{q \text{ copies}} = (V^*)^p \oplus V^q$$

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For $1 \leq i \leq p$ and $1 \leq j \leq q$, define $f_{ij} : (V^*)^p \oplus V^q \to \mathbb{C}$ by

$$f_{ij}(\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q) = \lambda_i(v_j).$$

Then by construction $f_{ij} \in \mathbb{C}[(V^*)^p \oplus V^q]$.

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In fact, $f_{ij} \in \mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$:

$$\begin{aligned} f_{ij}(g \cdot (\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q)) &= f_{ij}(g \cdot \lambda_1, \ldots, g \cdot \lambda_p, g \cdot v_1, \ldots, g \cdot v_q) \\ &= f_{ij}\left(\lambda_1 g^{-1}, \ldots, \lambda_p g^{-1}, g \cdot v_1, \ldots, g \cdot v_q\right) \\ &= \lambda_i\left(g^{-1} g\right) \cdot v_j \\ &= \lambda_i v_j = f_{ij}(\lambda_1, \ldots, \lambda_p, v_1, \ldots, v_q) \end{aligned}$$

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Theorem (FFT for $GL(V)$, Weyl 1939)

The basic invariants f_{ij} generate the invariant ring $\mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$ as a $\mathbb{C}$-algebra.

Weyls Second Fundamental Theorem

$S := \mathbb{C}[z_{ij} \mid 1 \le i \le p, 1 \le j \le q]$ polynomial ring in $p \times q$ variables

Consider the $\mathbb{C}$-algebra hom $\epsilon : S \to \mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$ given by

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Then, by the FFT,

$$S \xrightarrow{\epsilon} \mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)} \longrightarrow 0 \qquad \text{is exact.}$$

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Theorem (SFT for $GL(V)$, Weyl 1939)

The first syzygies of the invariant ring $\mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$ are generated by the $(k+1) \times (k+1)$ minors of the matrix $z = (z_{ij})$, where $k = \dim V$.

Remark: The minimal graded free resolution of $\mathbb{C}[(V^*)^p \oplus V^q]^{GL(V)}$ as an S-module was first computed by Lascoux in 1978 using an idea of Kempf.

Main Results

Main Results (Hunziker–M–Sepanski)

In the context of Hermann Weyl's fundamental theorems of invariant theory for $GL(V)$ we are able to:

- *Compute the syzygies of* **all** *modules of covariants* **uniformly**
- *Describe the differentials* **explicitly**

Will be similar for $O(V)$, $Sp(V)$.

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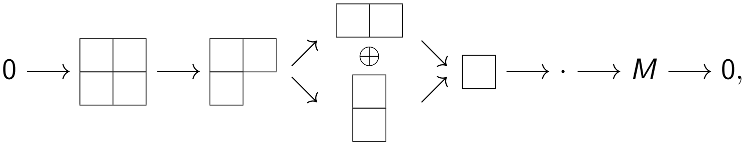
Visualization:



Today: Explain the basic set-up in the case $H = GL(V) = GL_k$ and give an example.

Notation for GL_k Reps

We make the following usual identifications for $H = GL_k$.

$$\{\text{poly irreps of } H\} \xleftrightarrow{bij} \{\lambda = (\lambda_1, \ldots, \lambda_k) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \cdots \geq \lambda_k \geq 0\}$$

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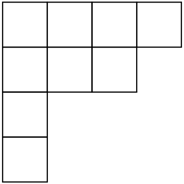
If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is such a sequence, it can be represented by a Young diagram with λ_1 boxes in the first row, λ_2 boxes in the second row, etc.

E.g. If $\lambda = (4, 3, 1, 1, 0)$, then $\lambda =$  .

We often just write the Young diagram λ for the corresponding representation $F_\lambda^{(k)}$.

Notation for GL_k Reps

For general irreps of $H = GL_k$ we have

$$\hat{H} = \{\text{irreps of } H\} \overset{bij}{\longleftrightarrow} \{\sigma = (\sigma_1, \dots, \sigma_k) \mid \sigma_i \in \mathbb{Z}, \sigma_1 \geq \cdots \geq \sigma_k\}$$

So, a general GL_k irrep is

$$\sigma = \underbrace{(n_1, \dots, n_i, 0, \dots, 0, -m_j, \dots, -m_1)}_{k}$$

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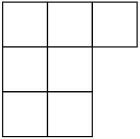
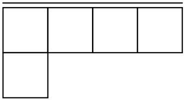
E.g. $\sigma = (3, 2, 2, 0, 0, -1, -4)$. We can associate to σ two Young diagrams:

$\sigma^+ = (3, 2, 2) =$  and $\sigma^- = (4, 1) =$ 

Back to Covariants

$$H = GL(V),\ W = \underbrace{V^* \oplus \cdots \oplus V^*}_{p \text{ copies}} \oplus \underbrace{V \oplus \cdots \oplus V}_{q \text{ copies}} = (V^*)^p \oplus V^q$$

Let $\Sigma = \left\{\sigma \in \hat{H} \mid \mathrm{Cov}_{GL(V)}\left((V^*)^p \oplus V^q, F_\sigma\right) \neq 0\right\}$.

Theorem (Howe, Kashiwara-Verge 1978)

$$\sigma \in \Sigma \iff \sigma = (n_1, \ldots, n_i, 0, \ldots, 0, -m_j, \ldots, -m_1) \ \textit{with}$$
$$0 \leq i \leq q,\ 0 \leq j \leq p$$

Furthermore, for each $\sigma \in \Sigma$ *we can associate a partition* $\lambda = \lambda(\sigma)$ *to* $\mathrm{Cov}_{GL(V)}\left((V^*)^p \oplus V^q, F_\sigma\right)$*, namely*

$$\lambda(\sigma) = (\underbrace{-k, \ldots, -k, -k - m_j, \ldots, -k - m_1}_{p}; \underbrace{n_1, \ldots, n_i, 0, \ldots, 0}_{q}),$$

and the map $\sigma \mapsto \lambda(\sigma)$ *is injective.*

Reduction of σ

$$\sigma = (n_1, \ldots, n_i, 0, \ldots, 0, -m_j, \ldots, -m_1)$$

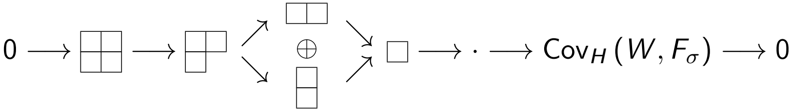
$$\rightsquigarrow$$

$$\lambda = \lambda(\sigma) = (\underbrace{-k, \ldots, -k, -k - m_j, \ldots, -k - m_1}_{p}; \underbrace{n_1, \ldots, n_i, 0, \ldots, 0}_{q})$$

$$\rightsquigarrow$$

$$(\lambda + \rho)' = (\underbrace{\_\_, \ldots, \_\_}_{p'}; \underbrace{\_\_, \ldots, \_\_}_{q'})$$

The terms in the resolution for $\mathrm{Cov}_{GL(V)}((V^*)^p \oplus V^q, F_\sigma)$ will be parametrized by all Young diagrams contained in a $p' \times q'$ rectangle.



Example

$\dim V = 4, \quad H = GL(V) = GL_4, \quad W = (V^*)^5 \oplus V^5$

$M = \mathrm{Cov}_{GL(V)}\left((V^*)^5 \oplus V^5, F_\sigma^{(4)}\right), \qquad p = q = 5, \quad k = 4.$

Let $\sigma = (3, 0, -1, 4)$.

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$$\sigma = (3, 0 - 1, -4)$$

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$$\lambda = \lambda(\sigma) = (\underbrace{-4, -4, -4, -4 - m_2, -4 - m_1}_{5}; \underbrace{n_1, 0, 0, 0, 0}_{5})$$

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Let $\rho = (p + q - 1, \ldots, 0) = (9, 8, 7, 6, 5, 4, 3, 2, 1, 0)$, so that

$$\lambda + \rho = (5, 4, 3, 1, -3; 7, 3, 2, 1, 0)$$

How to get from $\lambda + \rho$ to $(\lambda + \rho)'$?

Example

$\lambda + \rho = (5, 4, 3, 1, -3; 7, 3, 2, 1, 0)$
To get $(\lambda + \rho)'$:

(1) Delete repeats on left & right

$$\left(5, 4, \widehat{3}, \widehat{1}, -3; 7, \widehat{3}, 2, \widehat{1}, 0\right) = (5, 4, -3; 7, 2, 0)$$

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(2) Delete any numbers on left $<$ all numbers on right

$$\left(5, 4, \widehat{-3}; 7, 2, 0\right) = (5, 4; 7, 2, 0)$$

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$$\left(5, 4, \widehat{-3}; 7, 2, 0\right) = (5, 4; 7, 2, 0)$$

(3) Delete any numbers on right $>$ all numbers on left

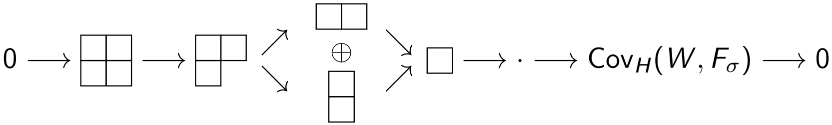
$$\left(5, 4; \widehat{7}, 2, 0\right) = (5, 4; 2, 0)$$

So $(\lambda + \rho)' = (5, 4; 2, 0)$, and $p' = q' = 2$.

Example

$$H = GL(V) = GL_4, \quad W = (V^*)^5 \oplus V^5, \quad \sigma = (3, 0, -1, -4)$$

$$\rightsquigarrow (\lambda + \rho)' = (5, 4; 2, 0), \quad p' = q' = 2$$



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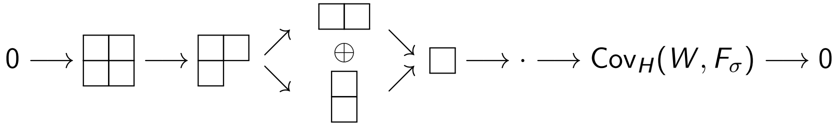

Starting with the box on the bottom left, label diagonally via successive differences in $(\lambda+\rho)' = (5, 4; 2, 0)$:

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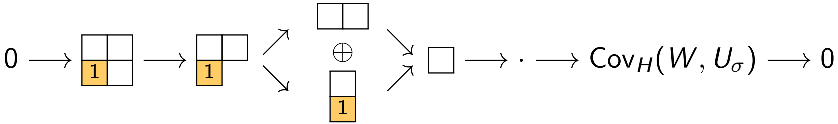

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$5 - 4 = 1$

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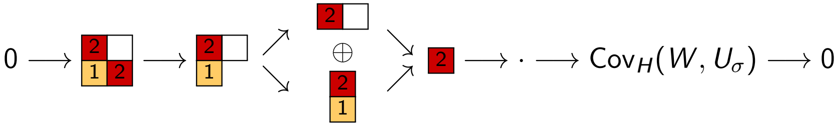

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$5 - 4 = 1$

$4 - 2 = 2$

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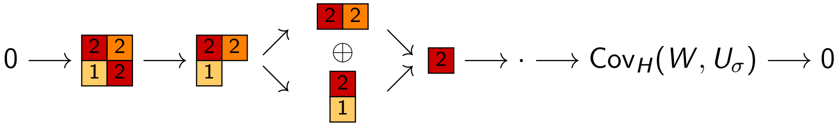


Starting with the box on the bottom left, label diagonally via successive differences in $(\lambda + \rho)' = (5, 4; 2, 0)$:

$5 - 4 = 1$

$4 - 2 = 2$

$2 - 0 = 2$

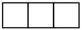
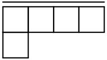
This “compressed” version needs to be unzipped.

Example

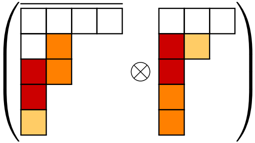
Unzip $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$:

Example

Unzip  :

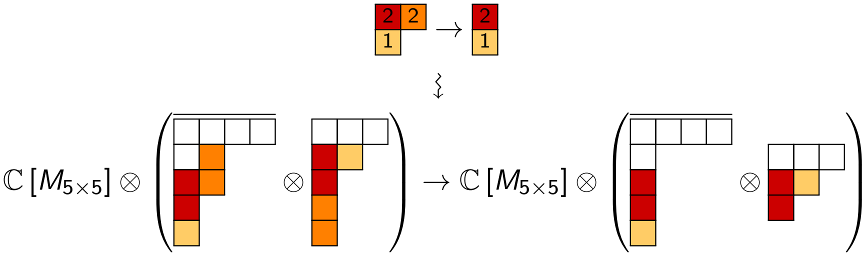
$\sigma = (3, 0, -1, -4) \rightsquigarrow$ $\sigma^+ = (3) =$ , $\sigma^- = (4, 1) =$ 

Add boxes to the columns of σ^- and σ^+ as prescribed by the colored diagram and its transpose, respectively.

$= \mathbb{C}[M_{5\times 5}] \otimes$ 

Example

Then the maps in the resolution look like



Here, each tableau on the left is losing 2 boxes.

Example

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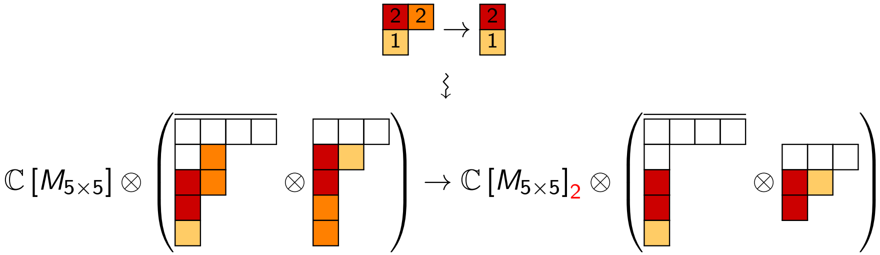

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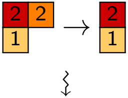
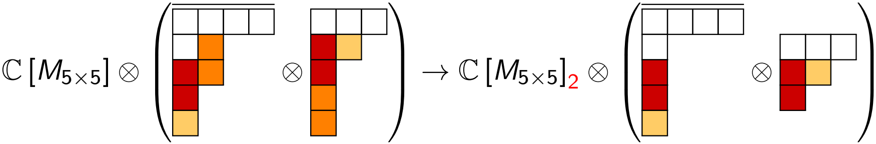
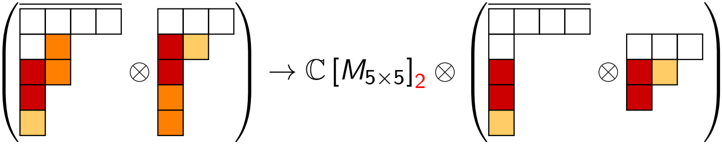
Then the maps in the resolution look like



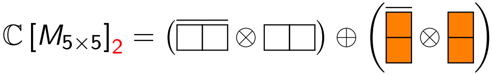


As an S-module map, it is enough to look at



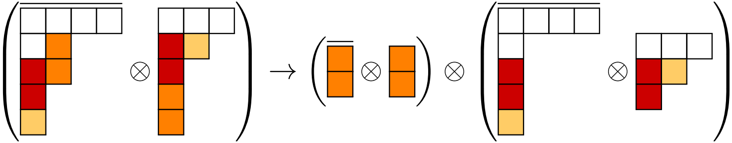
Example

As a $GL_5 \times GL_5$ rep, $\mathbb{C}[M_{5\times5}]_2$ decomposes into



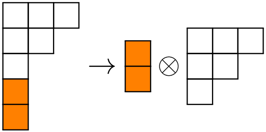
In general, $\mathbb{C}[M_{5\times5}]_d$ decomposes as above with all shapes that have d boxes and ≤ 5 rows.

Then we have

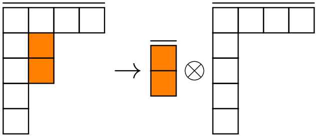


Example

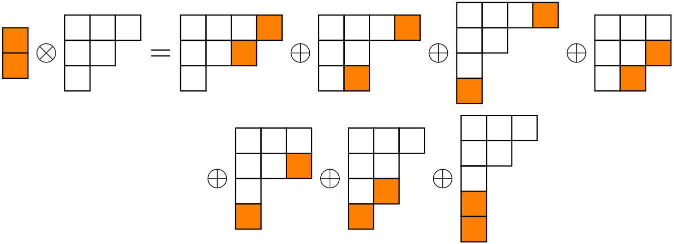
Computing the syzygies for $\mathrm{Cov}_{GL(V)}\left(\left(V^*\right)^5 \oplus V^5, F_\sigma^{(4)}\right)$ then comes down to finding S-module and $GL\left(V\right)$-equivariant **Pieri inclusions**, such as



and

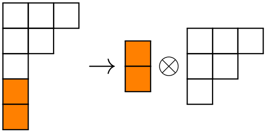


Pieri Rule:

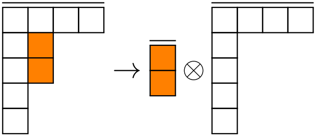


Example

Computing the syzygies for $\mathrm{Cov}_{GL(V)}\left(\left(V^*\right)^5 \oplus V^5, F_\sigma^{(4)}\right)$ then comes down to finding S-module and $GL(V)$-equivariant **Pieri inclusions**, such as



and



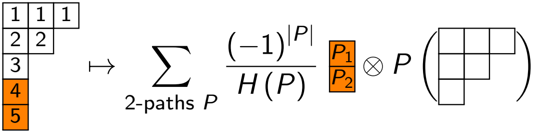
These Pieri inclusions were first given by Olver in his thesis (1982) via iterating one box removal, and made more explicit by Sam (2009) and Sam and Weyman (2012).

We give a general rule for removing m boxes.

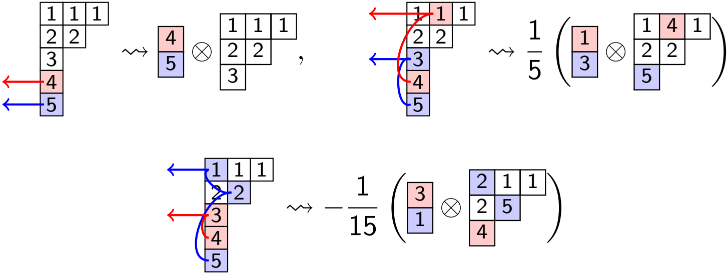
These maps are given by sums over certain paths in the tableaux.

Consider the two sides separately, then put them together.

Example



Terms in the sum look like



Example

$$
\begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 3 \\ *(orange) 4 \\ *(orange) 5 \end{ytableau}
\mapsto
\begin{ytableau} 4 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau}
- \begin{ytableau} 3 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 4 \end{ytableau}
- \frac{2}{3}\left(\begin{ytableau} 2 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 1 \\ 2 & 4 \\ 3 \end{ytableau} \right)
$$

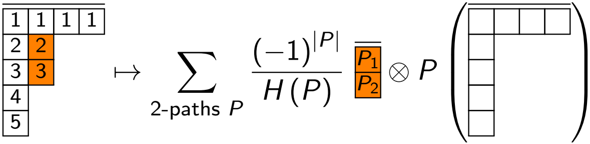
$$
+ \frac{2}{3}\left(\begin{ytableau} 2 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 1 \\ 2 & 3 \\ 4 \end{ytableau} \right)
- \frac{3}{5}\left(\begin{ytableau} 1 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 4 \\ 2 & 2 \\ 3 \end{ytableau} \right)
$$

$$
- \frac{6}{15}\left(\begin{ytableau} 1 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \\ 4 \end{ytableau} \right)
+ \frac{6}{15}\left(\begin{ytableau} 1 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau} \right)
$$

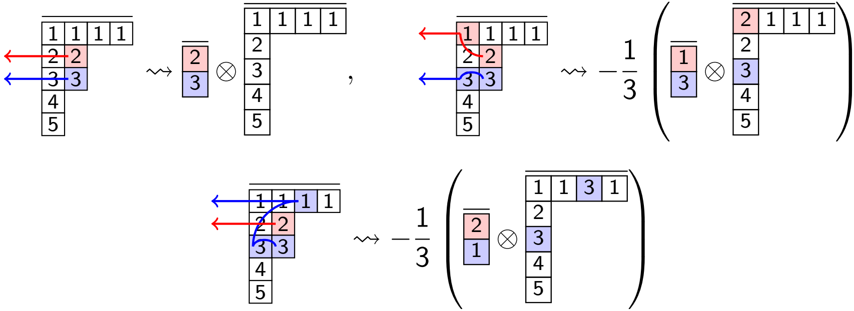
$$
+ \frac{3}{5}\left(\begin{ytableau} 1 \\ 5 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 3 \\ 2 & 2 \\ 4 \end{ytableau} \right)
+ \begin{ytableau} 3 \\ 4 \end{ytableau} \otimes \begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 5 \end{ytableau}
$$

$$
+ \cdots
$$

Example



Terms in the sum look like



Example

$$\overline{\begin{ytableau} 1 & 1 & 1 & 1 \\ 2 & *(orange) 2 \\ 3 & *(orange) 3 \\ 4 \\ 5 \end{ytableau}} \mapsto \quad 4 \left(\overline{\begin{ytableau} 2 \\ 3 \end{ytableau}} \otimes \overline{\begin{ytableau} 1 & 1 & 1 & 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{ytableau}} \right) - \frac{16}{3} \left(\overline{\begin{ytableau} 1 \\ 3 \end{ytableau}} \otimes \overline{\begin{ytableau} 1 & 1 & 1 & 2 \\ 2 \\ 3 \\ 4 \\ 5 \end{ytableau}} \right)$$

$$+ \frac{16}{3} \left(\overline{\begin{ytableau} 1 \\ 2 \end{ytableau}} \otimes \overline{\begin{ytableau} 1 & 1 & \, & 1 & 3 \\ 2 \\ 3 \\ 4 \\ 5 \end{ytableau}} \right)$$

Example

$$\overline{\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&\multicolumn{2}{c}{}\\\cline{1-2} 3&3&\multicolumn{2}{c}{}\\\cline{1-2} 4&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2&\multicolumn{1}{c}{}\\\cline{1-2} 3&\multicolumn{2}{c}{}\\\cline{1-1} 4&\multicolumn{2}{c}{}\\\cline{1-1} 5&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}$$

$$\mapsto$$

$$\sum_{\substack{\text{2-paths } \overline{P},\\ \text{2-paths } Q}} \frac{(-1)^{\left|\overline{P}\right|+|Q|}}{H\left(\overline{P}\right)H(Q)} \left(\overline{\begin{array}{|c|}\hline P_1\\\hline P_2\\\hline\end{array}} \otimes \begin{array}{|c|}\hline Q_1\\\hline Q_2\\\hline\end{array}\right) \otimes \left(\overline{P}\left(\overline{\begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline \ &\multicolumn{3}{c}{}\\\cline{1-1} \ &\multicolumn{3}{c}{}\\\cline{1-1} \ &\multicolumn{3}{c}{}\\\cline{1-1} \ &\multicolumn{3}{c}{}\\\cline{1-1}\end{array}}\right) \otimes Q\left(\begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline \ &\multicolumn{3}{c}{}\\\cline{1-1} \ &\multicolumn{3}{c}{}\\\cline{1-1} \ &\multicolumn{3}{c}{}\\\cline{1-1} \ &\multicolumn{3}{c}{}\\\cline{1-1}\end{array}\right)\right)$$

E.g., one term in the sum is

$$\frac{8}{5}\left(\overline{\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}} \otimes \begin{array}{|c|}\hline 1\\\hline 5\\\hline\end{array}\right) \otimes \left(\overline{\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&\multicolumn{3}{c}{}\\\cline{1-1} 3&\multicolumn{3}{c}{}\\\cline{1-1} 4&\multicolumn{3}{c}{}\\\cline{1-1} 5&\multicolumn{3}{c}{}\\\cline{1-1}\end{array}} \otimes \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&\multicolumn{1}{c}{}\\\cline{1-2} 4&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\right)$$

Example

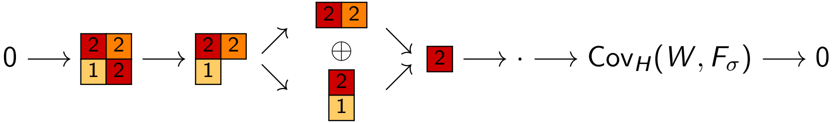
Still need $\overline{\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}} \otimes \begin{array}{|c|}\hline 1\\ \hline 5\\ \hline\end{array} \to \mathbb{C}\left[M_{5\times 5}\right]_2$

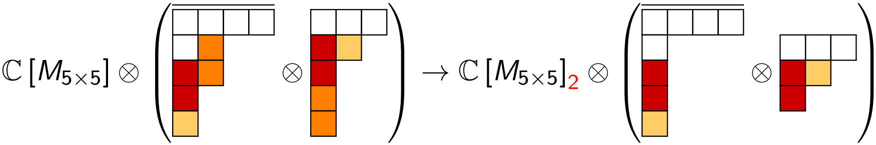
Via determinants:

$$\overline{\begin{array}{|c|}\hline {\color{red}2}\\ \hline {\color{red}3}\\ \hline\end{array}} \otimes \begin{array}{|c|}\hline {\color{blue}1}\\ \hline {\color{blue}5}\\ \hline\end{array} \quad \mapsto \quad \det \begin{array}{c} \begin{array}{ccccc} {\color{blue}1} & 2 & 3 & 4 & {\color{blue}5} \end{array} \\ \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ \mathbf{z_{21}} & z_{22} & z_{23} & z_{24} & \mathbf{z_{25}} \\ \mathbf{z_{31}} & z_{32} & z_{33} & z_{34} & \mathbf{z_{35}} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \end{pmatrix} \end{array} \begin{array}{c} 1 \\ {\color{red}2} \\ {\color{red}3} \\ 4 \\ 5 \end{array}$$

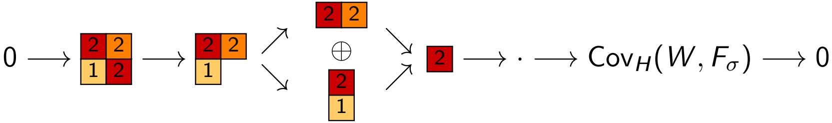
$$= z_{21}z_{35} - z_{31}z_{25}$$

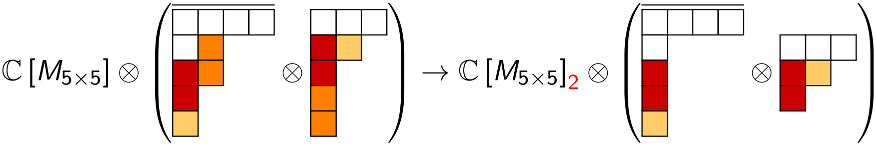
Example





Example





Thank You!